Pre- and Post-Lie Algebras: The Algebro-Geometric View

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Abstract We relate composition and substitution in pre- and post-Lie algebras to algebraic geometry. The Connes-Kreimer Hopf algebras and MKW Hopf algebras are then coordinate rings of the infinite-dimensional affine varieties consisting of series of trees, resp. Lie series of ordered trees. Furthermore we describe the Hopf algebras which are coordinate rings of the automorphism groups of these varieties, which govern the substitution law in pre- and post-Lie algebras.

1 Introduction

Pre-Lie algebras were first introduced in two different papers from 1963. Murray Gerstenhaber [13] studies deformations of algebras and Ernest Vinberg [29] problems in differential geometry. The same year John Butcher [2] published the first in a series of papers studying algebraic structures of numerical integration, culminating in his seminal paper [3] where B-series, the convolution product and the antipode of the Butcher–Connes–Kreimer Hopf algebra are introduced.

Post-Lie algebras are generalisations of pre-Lie algebras introduced in the last decade. Bruno Vallette [28] introduced the post-Lie operad as the Koszul dual of the commutative trialgebra operad. Simultaneously post-Lie algebras appear in the study of numerical integration on Lie groups and manifolds [21, 25]. In a differential geometric picture a pre-Lie algebra is the algebraic structure of the flat and torsion free connection on a locally Euclidean space, whereas post-Lie algebras appear naturally as the algebraic structure of the flat, constant torsion connection given by the Maurer–Cartan form on a Lie group [24]. Recently it is shown that the sections of an anchored vector bundle admits a post-Lie structure if and only if the bundle is an action Lie algebroid [22].

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B-series is a fundamental tool in the study of flow-maps (e.g. numerical integration) on Euclidean spaces. The generalised Lie-Butcher LB-series are combining B-series with Lie series and have been introduced for studying integration on Lie groups and manifolds.

In this paper we study B-series and LB-series from an algebraic geometry point of view. The space of B-series and LB-series can be defined as completions of the free pre- and post-Lie algebras. We study (L)B-series as an algebraic variety, where the coordinate ring has a natural Hopf algebra structure. In particular we are interested in the so-called substitution law. Substitutions for pre-Lie algebras were first introduced in numerical analysis [6]. The algebraic structure of pre-Lie substitutions and the underlying substitution Hopf algebra were introduced in [4]. For the post-Lie case, recursive formulae for substitution were given in [18]. However, the corresponding Hopf algebra of substitution for post-Lie algebras was not understood at that time.

In the present work we show that the algebraic geometry view gives a natural way to understand both the Hopf algebra of composition and the Hopf algebra of substitution for pre- and post-Lie algebras.

The paper is organised as follows. In Part 1 we study fundamental algebraic properties of the enveloping algebra of Lie-, pre-Lie and post-Lie algebras for the general setting that these algebras A are endowed with a decreasing filtration $A = A^1 \supseteq A^2 \supseteq \cdots$. This seems to be the general setting where we can define the exponential and logarithm maps, and define the (generalised) Butcher product for pre- and post-Lie algebras. Part 2 elaborates an algebraic geometric setting, where the pre- or post-Lie algebra forms an algebraic variety and the corresponding coordinate ring acquires the structure of a Hopf algebra. This yields the Hopf algebra of substitutions in the free post-Lie algebra. Finally, we provide a recursive formula for the coproduct in this substitution Hopf algebra.

Part 1: The Non-algebro Geometric Setting

In this part we have no type of finiteness condition on the Lie algebras, and pre- and post-Lie algebras. Especially in the first Sect. 2 the material will be largely familiar to the established reader.

2 The Exponential and Logarithm Maps for Lie Algebras

We work in the most general setting where we can define the exponential and logarithm maps. In Sect. 2.2 we assume the Lie algebra comes with a decreasing filtration, and is complete with respect to this filtration. We define the completed enveloping algebra, and discuss its properties. This is the natural general setting for the exponential and logarithm maps which we recall in Sect. 2.3.

2.1 The Euler Idempotent

The setting in this subsection is any Lie algebra L, finite or infinite dimensional over a field k of characteristic zero. Let $U(L)$ be its enveloping algebra. This is a Hopf algebra with unit η, counit ϵ and coproduct

$$\Delta : U(L) \to U(L) \otimes_k U(L)$$

defined by $\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1$ for any $\ell \in L$, and extended to all of $U(L)$ by requiring Δ to be an algebra homomorphism.

For any algebra A with multiplication map $\mu_A : A \otimes A \to A$, we have the convolution product $\star$ on $\mathrm{Hom}_k(U(L), A)$. For $f, g \in \mathrm{Hom}_k(U(L), A)$ it is defined as

$$f \star g = \mu_A \circ (f \otimes g) \circ \Delta_{U(L)}.$$

Let $\mathbf{1}$ be the identity map on $U(L)$, and $J = \mathbf{1} - \eta \circ \epsilon$. The *Eulerian idempotent* $e : U(L) \to U(L)$ is defined by

$$e = \log^{\star}(\mathbf{1}) = \log^{\star}(\eta \circ \epsilon + J) = J - \frac{J^{\star 2}}{2} + \frac{J^{\star 3}}{3} - \cdots .$$

Proposition 2.1 *The image of $e : U(L) \to U(L)$ is $L \subseteq U(L)$, and e is the identity restricted to L.*

Proof This is a special case of the canonical decomposition stated in 0.4.3 in [27]. See also Proposition 3.7, and part (i) of its proof in [27]. □

Let $\mathrm{Sym}^c(L)$ be the free cocommutative conilpotent coalgebra on L. It is the subcoalgebra of the tensor coalgebra $T^c(L)$ consisting of the symmetrized tensors

$$\sum_{\sigma \in S_n} l_{\sigma(1)} \otimes l_{\sigma(2)} \otimes \cdots l_{\sigma(n)} \in L^{\otimes n}, \quad l_1, \ldots l_n \in L. \tag{1}$$

The above proposition gives a linear map $U(L) \xrightarrow{e} L$. Since $U(L)$ is a cocommutative coalgebra, there is then a homomorphism of cocommutative coalgebras

$$U(L) \xrightarrow{\alpha} \mathrm{Sym}^c(L). \tag{2}$$

We now have the following strong version of the Poincaré-Birkhoff-Witt theorem.

Proposition 2.2 *The map $U(L) \xrightarrow{\alpha} Sym^c(L)$ is an isomorphism of coalgebras.*

In order to show this we expand more on the Euler idempotent.

Again for $l_1, \dots, l_n \in L$ denote by $(l_1, \dots, l_n)$ the symmetrized product in $U(L)$:

$$\frac{1}{n!}\sum_{\sigma \in S_n} l_{\sigma(1)} l_{\sigma(2)} \cdots l_{\sigma(n)}, \tag{3}$$

and let $U_n(L) \subseteq U(L)$ be the subspace generated by all these symmetrized products.

Proposition 2.3 *Consider the map given by convolution of the Eulerian idempotent:*

$$\frac{e^{\star p}}{p!} : U(L) \to U(L).$$

a. *The map above is zero on $U_q(L)$ when $q \neq p$ and the identity on $U_p(L)$.*
b. *The sum of these maps*

$$\exp^{\star p}(e) = \eta \circ \epsilon + e + \frac{e^{\star 2}}{2} + \frac{e^{\star 3}}{3!} + \cdots$$

is the identity map on $U(L)$. (Note that the map is well defined since the maps $e^{\star p}/p!$ vanish on any element in $U(L)$ for p sufficiently large.)

From the above we get a decomposition

$$U(L) = \bigoplus_{n \geq 0} U_n(L).$$

Proof This is the canonical decomposition stated in 0.4.3 in [27], see also Proposition 3.7 and its proof in [27]. □

Proof of Proposition 2.2 Note that since e vanishes on $U_n(L)$ for $n \geq 2$, by the way one constructs the map α, it sends the symmetrizer $(l_1, \dots, l_n) \in U_n(L)$ to the symmetrizer (3) in $\mathrm{Sym}^c_n(L)$. This shows α is surjective. But there is also a linear map, the surjective section $\beta : \mathrm{Sym}^c_n(L) \to U_n(L)$ sending the symmetrizer (3) to the symmetric product $(l_1, \dots, l_n)$. This shows that α must also be injective. □

2.2 Filtered Lie Algebras

Now the setting is that the Lie algebra L comes with a filtration

$$L = L^1 \supseteq L^2 \supseteq L^3 \supseteq \cdots$$

such that $[L^i, L^j] \subseteq L^{i+j}$. Examples of such may be derived from any Lie algebra over k:

1. The lower central series gives such a filtration with $L^2 = [L, L]$ and $L^{p+1} = [L^p, L]$.
2. The polynomials $L[h] = \oplus_{n \geq 1} Lh^n$.
3. The power series $L[\![h]\!] = \Pi_{n \geq 1} Lh^n$.

Let $\mathrm{Sym}_n(L)$ be the symmetric product of L, that is the natural quotient of $L^{\otimes n}$ which is the coinvariants $(L^{\otimes n})^{S_n}$ for the action of the symmetric group S_n. By the definition of $\mathrm{Sym}^c(L)$ in (1) there are maps

$$\mathrm{Sym}^c_n(L) \hookrightarrow L^{\otimes n} \rightarrow \mathrm{Sym}_n(L),$$

and the composition is a linear isomorphism. We get a filtration on $\mathrm{Sym}_n(L)$ by letting

$$F^p(\mathrm{Sym}_n(L)) = \sum_{i_1+\cdots+i_n \geq p} L^{i_1} \cdots L^{i_n}.$$

The filtration on L gives an associated graded Lie algebra $\mathrm{gr}\, L = \oplus_{i \geq 1} L_i / L_{i+1}$. The filtration on $\mathrm{Sym}_n(L)$ also induces an associated graded vector space.

Lemma 2.4 *There is an isomorphism of associated graded vector spaces*

$$Sym_n(gr\, L) \xrightarrow{\cong} gr\, Sym_n(L). \tag{4}$$

Proof Note first that there is a natural map (where d denotes the grading induced by the graded Lie algebra $\mathrm{gr}\, L$)

$$\mathrm{Sym}_n(\mathrm{gr}\, L)_d \rightarrow F^d\mathrm{Sym}_n(L)/F^{d+1}\mathrm{Sym}_n(L). \tag{5}$$

It is also clear by how the filtration is defined that any element on the right may be lifted to some element on the left, and so this map is surjective. We must then show that it is injective.

Choose splittings $L/L^{i+1} \xrightarrow{s_i} L$ of $L \rightarrow L/L^{i+1}$ for $i = 1, \ldots p$, and let $L_i = s_i(L^i/L^{i+1})$. Then we have a direct sum decomposition

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_p \oplus \cdots .$$

This gives an isomorphism $L \xrightarrow{\cong} \mathrm{gr}\, L$ which again gives a graded isomorphism

$$\mathrm{Sym}_n(L) \xrightarrow{\cong} \mathrm{Sym}_n(\mathrm{gr}\, L). \tag{6}$$

Since in general $\mathrm{Sym}_n(A \oplus B)$ is equal to $\oplus_i \mathrm{Sym}_i(A) \otimes \mathrm{Sym}_{n-i}(B)$ we get that

$$\mathrm{Sym}_n(L) = \oplus_{i_1,\ldots,i_p} S_{i_1}(L_1) \otimes \cdots \otimes S_{i_p}(L_p), \tag{7}$$

where we sum over all compositions where $\sum i_j = n$.

Claim

$$F^d S_n(L) = \oplus_{i_1,\ldots,i_p} S_{i_1}(L_1) \otimes \cdots \otimes S_{i_p}(L_p),$$

where we sum over all $\sum i_j = n$ and $\sum j \cdot i_j \geq d$.

This shows that the composition of (6) and (5) is an isomorphism. Therefore the map in (5) is an isomorphism.

Proof of Claim. Clearly we have an inclusion $\supseteq$. Conversely let $a \in F^d \mathrm{Sym}_n(L)$. Then a is a sum of products $a_{r_1} \cdots a_{r_q}$ where $a_{r_j} \in L^{r_j}$ and $\sum r_j \geq d$. But then each $a_{r_j} \in \oplus_{t \geq r_j} L_t$, and so by the direct sum decomposition in (7), each $a_{r_1} \cdots a_{r_q}$ lives in the right side of the claimed equality, and so does a. □

We have the enveloping algebra $U(L)$ and the enveloping algebra of the associated graded algebra $U(\mathrm{gr}\, L)$. The augmentation ideal $U(L)_+$ is the kernel $\ker U(L) \xrightarrow{\epsilon} k$ of the counit. The enveloping algebra $U(L)$ now gets a filtration of ideals by letting $F^1 = U(L)_+$ and

$$F^{p+1} = F^p \cdot U(L)_+ + (L^{p+1}),$$

where (L^{p+1}) is the ideal generated by L^{p+1}. This filtration induces again a graded algebra

$$\mathrm{gr}\, U(L) = \bigoplus_i F^i / F^{i+1}.$$

There is also another version, the graded product algebra, which we will encounter later

$$\mathrm{gr}^{\Pi} U(L) = \prod_i F^i / F^{i+1}.$$

Proposition 2.5 *The natural map of graded algebras*

$$U(gr\, L) \xrightarrow{\cong} gr\, U(L),$$

is an isomorphism.

Proof The filtrations on each $\mathrm{Sym}^c_n(L)$ induces a filtration on $\mathrm{Sym}^c(L)$. Via the isomorphism α of (2) and the explicit form given in the proof of Proposition 2.2 the filtrations on $U(L)$ and on $\mathrm{Sym}^c_n(L)$ correspond. Hence

$$\mathrm{gr}\,\alpha : \mathrm{gr}\,U(L) \xrightarrow{\cong} \mathrm{gr}\,\mathrm{Sym}^c_n(L)$$

is an isomorphism of vector spaces. There is also an isomorphism β and a commutative diagram

$$\begin{array}{ccc} U(\mathrm{gr}\,L) & \xrightarrow{\beta} & \mathrm{Sym}^c(\mathrm{gr}\,L) \\ \downarrow & & \downarrow \\ \mathrm{gr}\,U(L) & \xrightarrow{\mathrm{gr}\,\alpha} & \mathrm{gr}\,\mathrm{Sym}^c(L). \end{array}$$

By Lemma 2.4 the right vertical map is an isomorphism and so also the left vertical map. □

The cofiltration

$$\cdots \twoheadrightarrow U(L)/F^n \twoheadrightarrow U(L)/F^{n-1} \twoheadrightarrow \cdots$$

induces the completion

$$\hat{U}(L) = \varprojlim_p U(L)/F^p.$$

This algebra also comes with the filtration $\hat{F}^p$. Let $\hat{L} = \varprojlim_p L/L^p$.

Lemma 2.6 *The completed algebras are equal:*

$$\hat{U}(\hat{L}) = \hat{U}(L),$$

and so this algebra only depends on the completion $\hat{L}$.

Proof The natural map $L \to \hat{L}$ induces a natural map $U(L) \xrightarrow{\gamma} U(\hat{L})$. Since L and $\hat{L}$ have the same associated graded Lie algebras, the two downward maps in the commutative diagram

$$\begin{array}{ccc} \mathrm{gr}\,U(L) & \longrightarrow & \mathrm{gr}\,U(\hat{L}) \\ & \searrow \quad \swarrow & \\ & U(\mathrm{gr}\,L) & \end{array}$$

are isomorphisms, showing that the upper horizontal map is an isomorphism. But given the natural map γ this easily implies that the map of quotients

$$U(L)/F^{p+1}U(L) \xrightarrow{\gamma^p} U(\hat{L})/F^{p+1}U(\hat{L})$$

is an isomorphism, and so the completions are isomorphic. □

We denote the d'th graded part of the enveloping algebra $U(\mathrm{gr}\, L)$ by $U(\mathrm{gr}\, L)_d$. The following gives an idea of the "size" of $\hat{U}(L)$.

Lemma 2.7

$$gr^{\Pi}\hat{U}(L) = \hat{U}(gr\, L) = \prod_{d\in\mathbb{Z}} U(gr\, L)_d.$$

Proof The left graded product is

$$\mathrm{gr}^{\Pi}\hat{U}(L) = \prod_{p\geq 0} F^p/F^{p+1}.$$

But by Proposition 2.5 $F^p/F^{p+1} \cong U(\mathrm{gr}\, L)_p$ and so the above statement follows. □

Example 2.8 Let $V = \oplus_{i\geq 1} V_i$ be a graded vector space with V_i of degree i, and let $\mathrm{Lie}(V)$ be the free Lie algebra on V. It then has a grading $\mathrm{Lie}(V) = \oplus_{d\geq 1}\mathrm{Lie}(V)_d$ coming from the grading on V, and so a filtration $F^p = \oplus_{d\geq p}\mathrm{Lie}(V)_d$. The enveloping algebra $U(\mathrm{Lie}(V))$ is the tensor algebra $T(V)$. The completed enveloping algebra is

$$\hat{U}(\mathrm{Lie}(V)) = \hat{T}(V) := \prod_d T(V)_d.$$

Let L_p be the quotient L/L^{p+1}, which is a nilpotent filtered Lie algebra. We get enveloping algebras $U(L_p)$ with filtrations $F^jU(L_p)$ of ideals, and quotient algebras

$$U^j(L_p) = U(L_p)/F^{j+1}U(L_p).$$

Lemma 2.9

$$\hat{U}(L) = \varprojlim_{j,p} U^j(L_p).$$

Proof First note that if $j \leq p$ then $U^p(L_p) \twoheadrightarrow U^j(L_p)$ surjects. If $j \geq p$, then $U^j(L_j) \twoheadrightarrow U^j(L_p)$ surjects. Hence it is enough to show that the natural map

$$U(L)/F^{p+1} \to U(L_p)/F^{p+1}U(L_p) = U^p(L_p)$$

is an isomorphism. This follows since we have an isomorphism of associated graded vector spaces:

$$\begin{aligned}(\mathrm{gr}\,(U(L)/F^{p+1}))_{\leq p} &= (\mathrm{gr}\,U(L))_{\leq p} \cong U(\mathrm{gr}\,L)_{\leq p} \\ &= U(\mathrm{gr}\,L_p)_{\leq p} \cong (\mathrm{gr}\,U(L_p))_{\leq p} \\ &= (\mathrm{gr}\,U(L_p)/F^{p+1})_{\leq p}\end{aligned}$$

□

2.3 The Exponential and Logarithm

The coproduct Δ on $U(L)$ will send

$$F^p \xrightarrow{\Delta} 1 \otimes F^p + F^1 \otimes F^{p-1} + \cdots + F^p \otimes 1.$$

Thus we get a map

$$\hat{U}(L) \to U(L)/F^{2p-1} \xrightarrow{\Delta} U(L)/F^p \otimes U(L)/F^p.$$

Let

$$\hat{U}(L)\hat{\otimes}\hat{U}(L) := \varprojlim_p U(L)/F^p \otimes U(L)/F^p$$

be the completed tensor product We then get a *completed coproduct*

$$\hat{U}(L) \xrightarrow{\Delta} \hat{U}(L)\hat{\otimes}\hat{U}(L).$$

Note that the tensor product

$$\hat{U}(L) \otimes \hat{U}(L) \subseteq \hat{U}(L)\hat{\otimes}\hat{U}(L).$$

An element g of $\hat{U}(L)$ is *grouplike* if $\Delta(g) = g \otimes g$ in $\hat{U}(L) \otimes \hat{U}(L)$. We denote the set of grouplike elements by $G(\hat{U}(L))$. They are all of the form $1 + s$ where s is in the augmentation ideal

$$\hat{U}(L)_+ = \ker(\hat{U}(L) \xrightarrow{\epsilon} k).$$

The exponential map

$$\hat{U}(L)_+ \xrightarrow{exp} 1 + \hat{U}(L)_+$$

is given by

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

The logarithm map

$$1 + \hat{U}(L)_+ \xrightarrow{log} \hat{U}(L)_+$$

is defined by

$$\log(1 + s) = s - \frac{s^2}{2} + \frac{s^3}{3} - \cdots .$$

Proposition 2.10 *The maps*

$$\hat{U}(L)_+ \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + \hat{U}(L)_+$$

give inverse bijections. They restrict to inverse bijections

$$\hat{L} \underset{\log}{\overset{\exp}{\rightleftarrows}} G(\hat{U}(L))$$

between the completed Lie algebra and the grouplike elements.

Proof That $\log(\exp(x)) = x$ and $\exp(\log(1+s)) = 1+s$, are formal manipulations. If $\ell \in \hat{L}$ it is again a formal manipulation that

$$\Delta(\exp(\ell)) = \exp(\ell) \cdot \exp(\ell),$$

and so $\exp(\ell)$ is a grouplike element.

The maps exp and log can also be defined on the tensor products and give inverse bijections

$$\hat{U}(L)_+ \hat{\otimes} \hat{U}(L) + \hat{U}(L) \hat{\otimes} \hat{U}(L)_+ \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 \otimes 1 + \hat{U}(L)_+ \hat{\otimes} \hat{U}(L) + \hat{U}(L) \hat{\otimes} \hat{U}(L)_+ .$$

Now let $s \in G(\hat{U}(L))$ be a grouplike element. Since $\Delta = \Delta_{\hat{U}(L)}$ is an algebra homomorphism

$$\exp(\Delta(\log(s))) = \Delta(\exp(\log s)) = \Delta(s) = s \otimes s.$$

Since $1 \otimes s$ and $s \otimes 1$ are commuting elements we also have

$$\exp(\log(s) \otimes 1 + 1 \otimes \log(s)) = (\exp(\log(s)) \otimes 1) \cdot (1 \otimes \exp(\log(s))) = s \otimes s.$$

Taking logarithms of these two equations, we obtain

$$\Delta(\log(s)) = \log(s) \otimes 1 + 1 \otimes \log(s),$$

and so $\log(s)$ is in $\hat{L}$. □

3 Exponentials and Logarithms for Pre- and Post-Lie Algebras

For pre- and post-Lie algebras their enveloping algebra comes with two products $\bullet$ and $*$. This gives two possible exponential and logarithm maps. This is precisely the setting that enables us to define a map from formal vector fields to formal flows. It also gives the general setting for defining the Butcher product.

3.1 Filtered Pre- and Post-Lie Algebras

Given a linear binary operation on a k-vector space A

$$* : A \otimes_k A \to A$$

the associator is defined as:

$$a_*(x, y, z) = x * (y * z) - (x * y) * z.$$

Definition 3.1 A *post-Lie algebra* $(P, [,], \rhd)$ is a Lie algebra $(P, [,])$ together with a linear binary map $\rhd$ such that

- $x \rhd [y, z] = [x \rhd y, z] + [y, x \rhd z]$
- $[x, y] \rhd z = a_\rhd(x, y, z) - a_\rhd(y, x, z)$

It is then straightforward to verify that the following bracket

$$[\![x, y]\!] = x \triangleright y - y \triangleright x + [x, y]$$

defines another Lie algebra structure on P.

A *pre-Lie* algebra is a post-Lie algebra P such that bracket [,] is zero, so P with this bracket is the abelian Lie algebra.

Example 3.2 Let $\mathcal{X}\mathbb{R}^n$ be the vector fields on the manifold $\mathbb{R}^n$. It comes with the natural Levi-Cevita connection ∇. Write $f = \sum_{i=1}^n f^i \partial_i$ and $g = \sum_{i=1}^n g^i \partial_i$ for two vector fields, where $\partial_i = \partial / \partial x_i$. Let

$$f \triangleright g = \nabla_f g = \sum_{i,j} f^j (\partial_j g^i) \partial_i.$$

Then $\mathcal{X}\mathbb{R}^n$ is a pre-Lie algebra with this operation. Hence also a post-Lie algebra with trivial Lie-bracket [,] equal to zero.

Example 3.3 Let M be a manifold and $\mathcal{X}M$ the vector fields on M. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\lambda : \mathfrak{g} \to \mathcal{X}M$ be a morphism of Lie algebras. Denote by $\Omega^0(M, \mathfrak{g})$ the space of smooth maps $M \to \mathfrak{g}$. This is a Lie algebra by

$$[x, y](u) = [x(u), y(u)].$$

The vector fields $\mathcal{X}M$ act on the functions $\Omega^0(M, k)$ by differentiation: For $f \in \mathcal{X}M$ and $\phi \in \Omega^0(M, k)$ we get $f\phi \in \Omega^0(M, k)$. Hence $\mathcal{X}M$ acts on $\Omega^0(M, \mathfrak{g}) = \Omega^0(M, k) \otimes_k \mathfrak{g}$.

Now define the operation

$$\Omega^0(M, \mathfrak{g}) \times \Omega^0(M, \mathfrak{g}) \xrightarrow{\triangleright} \Omega^0(M, \mathfrak{g})$$
$$x \triangleright y \mapsto [u \mapsto (\lambda(x(u))y)(u)].$$

Then $\Omega^0(M, \mathfrak{g})$, [,], $\triangleright$ becomes a post-Lie algebra by [24, Prop.2.10].

If $G \times M \to M$ is an action of a Lie group G on M then for each $u \in M$ we get a map $G \to M$ and on tangent spaces $\mathfrak{g} \to T_u M$. This gives a map to the tangent bundle of M: $\mathfrak{g} \times M \to TM$ and map of Lie algebras $\mathfrak{g} \to \mathcal{X}M$. Hence in this setting we get by the above a post-Lie algebra $\Omega^0(M, \mathfrak{g})$.

If $M = G$ and $G \times G \to G$ is the Lie group operation, then $\Omega^0(G, \mathfrak{g})$ naturally identifies with the vector fields $\mathcal{X}G$ by left multiplication, and so these vector fields becomes a post-Lie algebra. In the special case that $G = \mathbb{R}^n$ with group operation $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sending $(a, b) \mapsto a + b$, we get the pre-Lie algebra of Example 3.2 above.

We now assume that P is a filtered post-Lie algebra: We have a decreasing filtration

$$P = P^1 \supseteq P^2 \supseteq \cdots,$$

such that

$$[P^p, P^q] \subseteq P^{p+q}, \quad P^p \rhd P^q \subseteq P^{p+q},$$

Then we will also have $[\![P^p, P^q]\!] \subseteq P^{p+q}$. If u and v are two elements of P such that $u - v \in P^{n+1}$, we say they are equal up to order n.

Again examples of this can be constructed for any post-Lie algebra over a field k by letting $P^1 = P$ and

$$P^{p+1} := P^p \rhd P + P \rhd P^p + [P, P^p].$$

Alternatively we may form the polynomials $P[h] = \oplus_{n\geq 1} Ph^n$, or the power series $P[\![h]\!] = \Pi_{n\geq 1} Ph^n$.

In [10] the enveloping algebra $U(P)$ of the post-Lie algebra was introduced. It is both the enveloping algebra for the Lie algebra $[,]$ and as such comes with associative product $\bullet$, and is the enveloping algebra for the Lie algebra $[\![,]\!]$ and as such comes with associative product $*$. The triangle product also extends to a product $\rhd$ on $U(P)$ but this is not associative.

3.2 The Map from Fields to Flows

By Example 3.2 above the formal power series of vector field $\mathcal{X}\mathbb{R}^n[\![h]\!]$ is a pre-Lie algebra, and from the last part of Example 3.3 we get a post-Lie algebra $\mathcal{X}G[\![h]\!]$ of series of vector fields. Using this perspective there are several natural ways to think of filtered post-Lie algebras and the related objects.

- The elements of P may be thought of as formal vector fields, in which case we write P_{field}.
- The grouplike elements of $\hat{U}(P)$ may be thought of as formal flows.
- The elements of P may be thought of as principal parts of formal flows, see below, in which case we write P_{flow}.

Let us explain how these are related. In the rest of this subsection we assume that $P = \hat{P}$ is complete with respect to the filtration. The exponential map

$$P_{field} \xrightarrow{\exp^*} \hat{U}(P) \tag{8}$$

sends a vector field to a formal flow, a grouplike element in $\hat{U}(P)$. (Note that the notion of a grouplike element in $\hat{U}(P)$ only depends on the shuffle coproduct.)

We may take the logarithm

$$G(\hat{U}(P)) \xrightarrow{\log^\bullet} P. \tag{9}$$

So if $B \in G(\hat{U}(P))$ we get $b = \log^\bullet(B)$. We think of b also as a formal flow, the *principal part* or *first order part* of the formal flow B. It determines B by $B = \exp^\bullet(b)$. Note that in (8) the exponential is with respect to the $*$ operation, while in (9) the logarithm is with respect to the $\bullet$ operation.

When P is a pre-Lie algebra A, then $\hat{U}(P)$ is the completed symmetric algebra $\widehat{\mathrm{Sym}}(A)$ and $\log^\bullet$ is simply the projection $\widehat{\mathrm{Sym}}(A) \to A$. If B is a Butcher series parametrized by forests (see Sect. 6.3), then b is the Butcher series parametrized by trees. Thus b determines the flow, but the full series B is necessary to compute pull-backs of functions along the flow.

We thus get a bijection

$$\Phi : P_{field} \xrightarrow{\log^\bullet \circ \exp^*} P_{flow} \tag{10}$$

which maps vector fields to principal part flows. This map is closely related to the *Magnus expansion* [8]. Magnus expresses the exact flow as $\exp^*(tv) = \exp^\bullet(\Phi(tv))$, from which a differential equation for $\Phi(tv)$ can be derived.

Example 3.4 Consider the manifold $\mathbb{R}^n$ and let $\mathcal{X}\mathbb{R}^n$ be the vector fields on $\mathbb{R}^n$. Let $f = \sum_{i\geq 0} f_i h^i$ on $\mathbb{R}^n$ be a power series of vector fields where each $f_i \in \mathcal{X}\mathbb{R}^n$. It induces the flow series $\exp^*(hf)$ in $\hat{U}(\mathcal{X}\mathbb{R}^n[[h]])$. Since $\mathcal{X}\mathbb{R}^n$ is a pre-Lie algebra, the completed enveloping algebra is $\widehat{\mathrm{Sym}}(\mathcal{X}\mathbb{R}^n[[h]])$. Thus the series

$$\exp^*(hf) = 1 + \sum_{i\geq d\geq 1} F_{i,d} h^i$$

where the $F_{i,d} \in \mathrm{Sym}_d(\mathcal{X}\mathbb{R}^n[[h]])$ are d'th order differential operators. (Note that the principal part b is the $d = 1$ part.) It determines a flow $\Psi_h^f : \mathbb{R}^n \to \mathbb{R}^n$ sending a point P to $P(h)$. For any smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ the *pullback* of ϕ along the flow is the composition $\phi \circ \Psi_h^f : \mathbb{R}^n \to \mathbb{R}$ and is given by

$$\exp^*(hf)\phi = 1 + \sum_{i\geq d\geq 1} F_{i,d}(\phi) h^i,$$

see [17, Section 4.1] or [23, Section 2.1]. In particular when ϕ is a coordinate function x_p we get the coordinate $x_p(h)$ of $P(h)$ as given by

$$x_p(h) = \exp^*(hf)x_p = \sum_{i\geq d\geq 1} F_{i,d} x_p h^i = x_p + \sum_i F_{i,1} x_p h^i$$

since higher derivatives of x_p vanish. This shows concretely geometrically why the flow is determined by its principal part.

For a given principal flow $b \in P_{flow}$ computing its inverse image by the map (10) above, which is the vector field $\log^* \circ \exp^\bullet(b)$ is called *backward error* in numerical analysis [14, 19].

For $a, a' \in P_{field}$ let

$$a \ast a' = \log^*(\exp^*(a) * \exp^*(a')),$$

a product which is computed using the Baker-Campbell-Hausdorff (BCH) formula for the Lie algebra $[\![,]\!]$. With this product P_{field} becomes a pro-unipotent group. Transporting this product to P_{flow} using the bijection Φ in (10), we get for $b, b' \in P_{flow}$ a product

$$b \sharp b' = \log^\bullet(\exp^\bullet(b) * \exp^\bullet(b')),$$

the *composition* product for principal flows.

Example 3.5 We continue Example 3.4. Let $g = \sum_{i \geq 0} g_i h^i$ be another power series of vector fields, $\exp^*(hg)$ its flow series, and $\Psi_h^g : \mathbb{R}^n \to \mathbb{R}^n$ the flow it determines. Let c be the principal part of $\exp^*(hg)$. The composition of the flows $\Psi_h^g \circ \Psi_h^f$ is the flow sending ϕ to

$$\exp^*(hg)(\exp^*(hf)\phi) = (\exp^*(hg) * \exp^*(hf))\phi.$$

The principal part of the composed flow is

$$\log^\bullet(\exp^*(hg) * \exp^*(hf)) = \log^\bullet(\exp^\bullet(c) * \exp^\bullet(b)) = c \sharp b,$$

the Butcher product of c and b.

Denote by $\diamond$ the product in P_{flow} given by the BCH-formula for the Lie bracket $[,]$,

$$x \diamond y := \log^\bullet(\exp^\bullet(x) \bullet \exp^\bullet(y)).$$

Proposition 3.6 *For x, y in the post-Lie algebra P_{flow} we have*

$$x \sharp y = x \diamond \left(\exp^\bullet(x) \rhd y\right).$$

Proof From [10, Prop.3.3] the product $A * B = \sum_{\Delta(A)} A_{(1)}(A_{(2)} \rhd B)$. Since $\exp^\bullet(x)$ is a group-like element it follows that:

$$\exp^\bullet(x) * \exp^\bullet(y) = \exp^\bullet(x) \bullet \left(\exp^\bullet(x) \rhd \exp^\bullet(y)\right).$$

By [10, Prop.3.1] $A \rhd BC = \sum_{\Delta(A)} (A_{(1)} \rhd B)(A_{(2)} \rhd C)$ and so again using that $\exp^\bullet(x)$ is group-like and the expansion of $\exp^\bullet(y)$:

$$\exp^\bullet(x) \rhd \exp^\bullet(y) = \exp^\bullet\left(\exp^\bullet(x) \rhd y\right).$$

Hence

$$x \sharp y = \log^\bullet\Big(\exp^\bullet(x) \bullet \big(\exp^\bullet(x) \rhd \exp^\bullet(y)\big)\Big) = \log^\bullet\left(\exp^\bullet(x) \bullet \exp^\bullet(\exp^\bullet(x) \rhd y)\right).$$

$\square$

In the pre-Lie case $[,] = 0$, therefore $\diamond = +$ and we obtain the formula derived in [9]

$$x \sharp y = x + \exp^\bullet(x) \rhd y.$$

3.3 Substitution

Let $\mathrm{End}_{\mathrm{postLie}}(P) = \mathrm{Hom}_{\mathrm{postLie}}(P, P)$ be the endomorphisms of P as a post-Lie algebra. (In the special case that P is a pre-Lie algebra, this is simply the endomorphisms of P as a pre-Lie algebra.) It is a monoid, but not generally a vector space. It acts on the post-Lie algebra P.

Since the action respects the brackets $[,]$, $[\![,]\!]$ and $\rhd$, it also acts on the enveloping algebra $U(P)$ and its completion $\hat{U}(P)$, and respects the products $*$ and $\bullet$. Hence the exponential maps $\exp^*$ and $\exp^\bullet$ are equivariant for this action, and similarly the logarithms $\log^*$ and $\log^\bullet$. So the formal flow map

$$\Phi : P_{\text{field}} \longrightarrow P_{\text{flow}}$$

is equivariant for the action. The action on P_{flow} (which is technically the same as the action on P_{field}), is called *substitution* and is usually studied in a more specific context, as we do in Sect. 7. An element $\phi \in \mathrm{End}_{\mathrm{postLie}}(P)$ comes from sending a field f to a perturbed field f', and one then sees how this affects the exact flow or approximate flow maps given by numerical algorithms.

Part 2: The Algebraic Geometric Setting

In this part we have certain finiteness assumptions on the Lie algebras and pre- and post-Lie algebras, and so may consider them and binary operations on them in the setting of varieties. The first three subsections of the next Sect. 4 will be quite familiar to the reader who knows basic algebraic geometry.

4 Affine Varieties and Group Actions

We assume the reader is familiar with basic algebraic geometry of varieties and morphisms, like presented in [16, Chap.1] or [7, Chap.1,5]. We nevertheless briefly recall basic notions. A notable and not so standard feature is that we in the last subsection define infinite dimensional varieties and morphisms between them.

4.1 Basics on Affine Varieties

Let k be a field and $S = k[x_1, \ldots, x_n]$ the polynomial ring. The *affine* n*-space* is

$$\mathbb{A}^n_k = \{(a_1, \ldots, a_n) \mid a_i \in k\}.$$

An ideal $I \subseteq S$ defines an *affine variety* in $\mathbb{A}^n_k$:

$$X = \mathcal{Z}(I) = \{p \in \mathbb{A}^n_k \mid f(p) = 0, \text{ for } f \in I\}.$$

Given an affine variety $X \subseteq \mathbb{A}^n_k$, its associated ideal is

$$\mathcal{I}(X) = \{f \in S \mid f(p) = 0, \text{ for } p \in X\}.$$

Note that if $X = \mathcal{Z}(I)$ then $I \subseteq \mathcal{I}(X)$, and $\mathcal{I}(X)$ is the largest ideal defining the variety X. The correspondence

$$\text{ideals in } k[x_1, \ldots, x_n] \underset{\mathcal{I}}{\overset{\mathcal{Z}}{\rightleftarrows}} \text{subsets of } \mathbb{A}^n_k$$

is a Galois connection. Thus we get a one-to-one correspondence

$$\text{image of } \mathcal{I} \overset{1-1}{\longleftrightarrow} \underset{= \text{ varieties in } \mathbb{A}^n_k}{\text{image of } \mathcal{Z}} .$$

Remark 4.1 When the field k is algebraically closed, Hilbert's Nullstellensatz says that the image of $\mathcal{I}$ is precisely the radical ideals in the polynomial ring. In general however the image of $\mathcal{I}$ is only contained in the radical ideals.

The *coordinate ring* of a variety X is the ring $A(X) = k[x_1, \ldots, x_n]/\mathcal{I}(X)$. A morphism of affine varieties $f : X \to Y$ where $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$ is a a map sending a point $\mathbf{a} = (a_1, \ldots, a_n)$ to a point $(f_1(\mathbf{a}), \ldots, f_m(\mathbf{a}))$ where the f_i are polynomials in S. This gives rise to a homomorphism of coordinate rings

$$\begin{aligned} f^\sharp : A(Y) &\longrightarrow A(X) \\ \overline{y_i} &\longrightarrow f_i(\overline{\mathbf{x}}), \quad i = 1, \ldots, m \end{aligned}$$

In fact this is a one-one correspondence:

$$\{\text{morphisms } f : X \to Y\} \quad \overset{1-1}{\longleftrightarrow} \quad \{\text{algebra homomorphisms } f^\sharp : A(Y) \to A(X)\}.$$

The zero-dimensional affine space $\mathbb{A}^0_k$ is simply a point, and its coordinate ring is k. Therefore to give a point $p \in \mathbb{A}^n_k$ is equivalent to give an algebra homomorphism $k[x_1, \ldots, x_n] \to k$.

Remark 4.2 We may replace k by any commutative ring $\Bbbk$. The affine space $\mathbb{A}^n_{\Bbbk}$ is then $\Bbbk^n$. The coordinate ring of this affine space is $\Bbbk[x_1, \ldots, x_n]$. A point $p \in \mathbb{A}^n_{\Bbbk}$ still corresponds to an algebra homomorphism $\Bbbk[x_1, \ldots, x_n] \to \Bbbk$. Varieties in $\mathbb{A}^n_{\Bbbk}$ may be defined in the same way, and there is still a Galois connection between ideals in $\Bbbk[x_1, \ldots, x_n]$ and subsets of $\mathbb{A}^n_{\Bbbk}$, and a one-one correspondence between morphisms of varieties and coordinate rings.

The affine space $\mathbb{A}^n_k$ comes with the *Zariski topology*, whose closed sets are the affine varieties in $\mathbb{A}^n_k$ and whose open sets are the complements of these. This induces also the Zariski topology on any affine subvariety X in $\mathbb{A}^n_k$.

If X and Y are affine varieties in $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$ respectively, their product $X \times Y$ is an affine variety in $\mathbb{A}^{n+m}_k$ whose ideal is the ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by $\mathcal{I}(X) + \mathcal{I}(Y)$. Its coordinate ring is

$$A(X \times Y) = A(X) \otimes_k A(Y).$$

If A is a ring and $f \neq 0$ in A, we have the localized ring A_f whose elements are all a/f^n where $a \in A$. Two such elements a/f^n and b/f^m are equal if $f^k(f^m a - f^n b) = 0$ for some k. If A is an integral domain, this is equivalent to $f^m a - f^n b = 0$. Note that the localization A_f is isomorphic to the quotient ring $A[x]/(xf - 1)$. Hence if A is a finitely generated k-algebra, A_f is also a finitely generated k-algebra. A consequence of this is the following: Let X be an affine variety in $\mathbb{A}^n_k$ whose ideal is $I = \mathcal{I}(X)$ contained in $k[x_1, \ldots, x_n]$, and let f be a polynomial function. The open subset

$$D(f) = \{p \in X \mid f(p) \neq 0\} \subseteq X$$

is then in bijection to the variety $X' \in \mathbb{A}^{n+1}_k$ defined by the ideal $I + (x_{n+1} f - 1)$. This bijection is actually a homeomorphism in the Zariski topology. The coordinate ring

$$A(X') = A(X)[x_{n+1}]/(x_{n+1} f - 1) \cong A(X)_f.$$

Hence we identify A_f as the coordinate ring of the open subset $D(f)$ and can consider $D(f)$ as an affine variety. Henceforth we shall drop the adjective affine for a variety, since all our varieties will be affine.

4.2 Coordinate Free Descriptions of Varieties

For flexibility of argument, it may be desirable to consider varieties in a coordinate free context.

Let V and W be dual finite dimensional vector spaces. So $V = \mathrm{Hom}_k(W, k) = W^*$, and then W is naturally isomorphic to $V^* = (W^*)^*$. We consider V as an affine space (this means that we are forgetting the structure of vector space on V). Its coordinate ring is the symmetric algebra $\mathrm{Sym}(W)$. Note that any polynomial $f \in \mathrm{Sym}(W)$ may be evaluated on any point $\mathbf{v} \in V$, since $\mathbf{v} : W \to k$ gives maps $\mathrm{Sym}_d(W) \to \mathrm{Sym}_d(k) = k$ and thereby a map $\mathrm{Sym}(W) = \oplus_d \mathrm{Sym}_d(W) \to k$.

Given an ideal I in $\mathrm{Sym}(W)$, the associated affine variety is

$$X = \{\mathbf{v} \in V \mid f(\mathbf{v}) = 0, \text{ for } f \in I\} \subseteq V.$$

Given a variety $X \subseteq V$ we associate the ideal

$$\mathcal{I}(X) = \{f \in \mathrm{Sym}(W) \mid f(\mathbf{v}) = 0, \text{ for } \mathbf{v} \in X\} \subseteq \mathrm{Sym}(W).$$

The coordinate ring of X is $A(X) = \mathrm{Sym}(W)/\mathcal{I}(X)$.

Let W^1 and W^2 be two vector spaces, with dual spaces V^1 and V^2. A map $f : X^1 \to X^2$ between varieties in these spaces is a map which is given by polynomials once a coordinate system is fixed for V^1 and V^2. Such a map then gives a homomorphism of coordinate rings $f^\sharp : \mathrm{Sym}(W^2)/I(X^2) \to \mathrm{Sym}(W^1)/I(X^1)$, and this gives a one-one correspondence between morphisms f between X^1 and X^2 and algebra homomorphisms $f^\sharp$ between their coordinate rings.

4.3 Affine Spaces and Monoid Actions

The vector space of linear operators on V is denoted $\mathrm{End}(V)$. It is an affine space with $\mathrm{End}(V) \cong \mathbb{A}_k^{n\times n}$, and with coordinate ring $\mathrm{Sym}(\mathrm{End}(V)^*)$. We then have an action

$$\begin{aligned} \mathrm{End}(V) \times V &\to V \\ (\phi, v) &\mapsto \phi(v). \end{aligned} \tag{11}$$

This is a morphism of varieties. Explicitly, if V has basis $e_1, \ldots, e_n$ an element in $\mathrm{End}(V)$ may be represented by a matrix A and the map is given by:

$$(A, (v_1, \ldots, v_n)^t) \mapsto A \cdot (v_1, \ldots, v_n)^t,$$

which is given by polynomials.

The morphism of varieties (11) then corresponds to the algebra homomorphism on coordinate rings

$$\mathrm{Sym}(V^*) \to \mathrm{Sym}(\mathrm{End}(V)^*) \otimes_k \mathrm{Sym}(V^*).$$

With a basis for V, the coordinate ring $\mathrm{Sym}(\mathrm{End}(V)^*)$ is isomorphic to the polynomial ring $k[t_{ij}]_{i,j=1,\dots,n}$, where the t_{ij} are coordinate functions on $\mathrm{End}(V)$, and the coordinate ring $\mathrm{Sym}(V^*)$ is isomorphic to $k[x_1, \dots, x_n]$ where the x_i are coordinate functions on V. The map above on coordinate rings is then given by

$$x_i \mapsto \sum_j t_{ij} x_j.$$

We may also consider the set $\mathrm{GL}(V) \subseteq \mathrm{End}(V)$ of invertible linear operators. This is the open subset $D(\det(t_{ij}))$ of $\mathrm{End}(V)$ defined by the nonvanishing of the determinant. Hence, fixing a basis of V, its coordinate ring is the localized ring $k[t_{ij}]_{\det((t_{ij}))}$, by the last part of Sect. 4.1. The set $\mathrm{SL}(V) \subseteq \mathrm{End}(V)$ are the linear operators with determinant 1. This is a closed subset of $\mathrm{End}(V)$ defined by the polynomial equation $\det((t_{ij})) - 1 = 0$. Hence the coordinate ring of $\mathrm{SL}(V)$ is the quotient ring $k[t_{ij}]/(\det((t_{ij})) - 1)$.

Now given an affine monoid variety M, that is an affine variety with a product morphism $\mu : M \times M \to M$ which is associative and unital. Then we get an algebra homomorphism of coordinate rings

$$A(M) \xrightarrow{\Delta} A(M) \otimes_k A(M).$$

Since the following diagram commutes

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mu \times \mathbf{1}} & M \times M \\ {\scriptstyle \mathbf{1} \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\mu} & M, \end{array}$$

we get a commutative diagram of coordinate rings:

$$\begin{array}{ccc} A(M) \otimes_k A(M) \otimes_k A(M) & \xleftarrow[\Delta \otimes \mathbf{1}]{} & A(M) \otimes A(M) \\ {\scriptstyle \mathbf{1} \otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ A(M) \otimes_k A(M) & \xleftarrow{\Delta} & A(M). \end{array}$$

The zero-dimensional affine space $\mathbb{A}^0_k$ is simply a point, and its coordinate ring is k. A character on $A(M)$ is an algebra homomorphism $A(M) \to k$. On varieties this gives a morphism $P = \mathbb{A}_k^0 \to M$, or a point in the monoid variety. In particular the unit in M corresponds to a character $A(M) \xrightarrow{\epsilon} k$, the counit. Thus the algebra $A(M)$ with Δ and ϵ becomes a bialgebra.

The monoid may act on a variety X via a morphism of varieties

$$M \times X \to X. \tag{12}$$

On coordinate rings we get a homomorphism of algebras,

$$A(X) \to A(M) \otimes_k A(X), \tag{13}$$

making $A(X)$ into a comodule algebra over the bialgebra $A(M)$.

In coordinate systems the morphism (12) may be written:

$$(m_1, \ldots, m_r) \times (x_1, \ldots, x_n) \mapsto (f_1(\mathbf{m}, \mathbf{x}), f_2(\mathbf{m}, \mathbf{x}), \ldots).$$

If X is an affine space V and the action comes from a morphism of monoid varieties $M \to \text{End}(V)$, the action by M is linear on V. Then $f_i(\mathbf{m}, \mathbf{v}) = \sum_j f_{ij}(\mathbf{m}) v_j$. The homomorphism on coordinate rings (recall that $V = W^*$)

$$\text{Sym}(W) \to A(M) \otimes_k \text{Sym}(W)$$

is then induced from a morphism

$$W \to A(M) \otimes_k W$$
$$x_j \mapsto \sum_i f_{ij}(\mathbf{u}) \otimes_k x_i$$

where the x_j's are the coordinate functions on V and $\mathbf{u}$ are the coordinate functions on M.

We can also consider an affine group variety G with a morphism $G \to GL(V)$ and get a group action $G \times V \to V$. The inverse morphism for the group, induces an antipode on the coordinate ring $A(G)$ making it a commutative Hopf algebra.

4.4 Infinite Dimensional Affine Varieties and Monoid Actions

The infinite dimensional affine space $\mathbb{A}_k^\infty$ is $\prod_{i \geq 1} k$. Its elements are infinite sequences $(a_1, a_2, \ldots)$ where the a_i are in k. Its coordinate ring is the polynomial ring in infinitely many variables $S = k[x_i, i \in \mathbb{N}]$.

An ideal I in S, defines an affine variety

$$X = V(I) = \{\mathbf{a} \in \mathbb{A}_k^\infty \mid f(\mathbf{a}) = 0, \text{ for } f \in I\}.$$

Note that a polynomial f in S always involves only a finite number of the variables, so the evaluation $f(\mathbf{a})$ is meaningful. Given an affine variety X, let its ideal be:

$$\mathcal{I}(X) = \{f \in S \mid f(\mathbf{a}) = 0 \text{ for } \mathbf{a} \in X\}.$$

The coordinate ring $A(X)$ of X is the quotient ring $S/\mathcal{I}(X)$. The affine subvarieties of $\mathbb{A}_k^\infty$ form the closed subsets in the Zariski topology on $\mathbb{A}_k^\infty$, and this then induces the Zariski topology on any subvariety of $\mathbb{A}_k^\infty$.

A morphism $f : X \to Y$ of two varieties, is a map such that $f(\mathbf{a}) = (f_1(\mathbf{a}), f_2(\mathbf{a}), \ldots)$ where each f_i is a polynomial function (and so involves only a finite number of the coordinates of $\mathbf{a}$).

Letting $k[y_i, i \in \mathbb{N}]$ be the coordinate ring of affine space where Y lives, we get a morphism of coordinate rings

$$\begin{aligned} f^\sharp : A(Y) &\to A(X) \\ \overline{y_i} &\mapsto f_i(\overline{\mathbf{x}}) \end{aligned}$$

This gives a one-one correspondence

$$\{\text{morphisms } f : X \to Y\} \quad \leftrightarrow \quad \{\text{algebra homomorphisms } f^\sharp : A(Y) \to A(X)\}.$$

For flexibility of argument, it is desirable to have a coordinate free definition of these varieties also. The following includes then both the finite and infinite-dimensional case in a coordinate free way.

Let W be a vector space with a countable basis. We get the symmetric algebra $\mathrm{Sym}(W)$. Let $V = \mathrm{Hom}_k(W, k)$ be the dual vector space, which will be our affine space. Given an ideal I in $\mathrm{Sym}(W)$, the associated affine variety is

$$X = V(I) = \{\mathbf{v} \in V \mid f(\mathbf{v}) = 0, \text{ for } f \in I\}.$$

The evaluation of f on $\mathbf{v}$ is here as explained in Sect. 4.2. Given a variety X we associate the ideal

$$\mathcal{I}(X) = \{f \in \mathrm{Sym}(W) \mid f(\mathbf{v}) = 0, \text{ for } \mathbf{v} \in X\}.$$

Its coordinate ring is $A(X) = \mathrm{Sym}(W)/\mathcal{I}(X)$. We shall shortly define morphism between varieties. In order for these to be given by polynomial maps, we will need filtrations on our vector spaces. Given a filtration by finite dimensional vector spaces

$$\langle 0 \rangle = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W.$$

On the dual space V we get a decreasing filtration by $V^i = \ker((W)^* \to (W_{i-1})^*)$. The affine variety $V/V^i \cong (W_{i-1})^*$ has coordinate ring $\mathrm{Sym}(W_{i-1})$. If X is a variety in V its image X_i in the finite affine space V/V^i need not be Zariski closed. Let $\overline{X_i}$ be its closure. This is an affine variety in V/V^i whose ideal is $\mathcal{I}(X) \cap \mathrm{Sym}(W_{i-1})$.

A map $f : X_1 \to X_2$ between varieties in these spaces is a *morphism of varieties* if there exists decreasing filtrations

$$V_1 = V_1^1 \supseteq V_1^2 \supseteq \cdots, \quad V_2 = V_2^1 \supseteq V_2^2 \supseteq \cdots$$

with finite dimensional quotient spaces, such that for any i we have a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow & & \downarrow \\ \overline{X_{1,i}} & \longrightarrow & \overline{X_{2,i}} \end{array}$$

and the lower map is a morphism between varieties in V_1/V_1^i and V_2/V_2^i.

We then get a homomorphisms of coordinate rings

$$f_i^\sharp : \mathrm{Sym}(W_i^2)/\mathcal{I}(X_{2,i}) \to \mathrm{Sym}(W_i^1)/\mathcal{I}(X_{1,i}), \tag{14}$$

and the direct limit of these gives a homomorphism of coordinate rings

$$f^\sharp : \mathrm{Sym}(W^2)/\mathcal{I}(X_2) \to \mathrm{Sym}(W^1)/\mathcal{I}(X_1). \tag{15}$$

Conversely given an algebra homomorphism $f^\sharp$ above. Let

$$W_1^2 \subseteq W_2^2 \subseteq W_3^2 \subseteq \cdots$$

be a filtration. Write $W^1 = \oplus_{i \in \mathbb{N}} k w_i$ in terms of a basis. The image of W_i^2 will involve only a finite number of the w_i. Let W_i^1 be the f.d. subvector space generated by these w_i. Then we get maps (14), giving morphisms

$$\begin{array}{ccc} \overline{X_{1,i+1}} & \longrightarrow & \overline{X_{2,i+1}} \\ \downarrow & & \downarrow \\ \overline{X_{1,i}} & \longrightarrow & \overline{X_{2,i}}. \end{array}$$

In the limit we then get a morphism of varieties $f : X_1 \to X_2$. This gives a one-one correspondence between morphisms of varieties $f : X_1 \to X_2$ and algebra homomorphisms $f^\sharp$.

Let X^1 and X^2 be varieties in the affine spaces V^1 and V^2. Their product $X^1 \times X^2$ is a variety in the affine space $V^1 \times V^2$ which is the dual space of $W^1 \oplus W^2$. Its coordinate ring is $A(X^1 \times X^2) = A(X^1) \otimes_k A(X^2)$.

If M is an affine monoid variety (possibly infinite dimensional) its coordinate ring $A(M)$ becomes a commutative bialgebra. If M is an affine group variety, then $A(M)$ is a Hopf algebra. We can again further consider an action on the affine space

$$M \times V \to V.$$

It corresponds to a homomorphism of coordinate rings

$$\mathrm{Sym}(W) \to A(M) \otimes_k \mathrm{Sym}(W),$$

making $\mathrm{Sym}(W)$ into a comodule algebra over $A(M)$. If the action by M is linear on V, the algebra homomorphism above is induced by a linear map $W \to A(M) \otimes_k W$.

5 Filtered Algebras with Finite Dimensional Quotients

In this section we assume the quotients $L_p = L/L^{p+1}$ from Sect. 2.2 are finite dimensional vector spaces. This enables us to define the dual Hopf algebra $U^c(K)$ of the enveloping algebra $U(L)$. This Hopf algebra naturally identifies as the coordinate ring of the completed Lie algebra $\hat{L}$. In Sect. 5.3 the Baker-Campbell-Hausdorff product on the variety L is shown to correspond to the natural coproduct on the dual Hopf algebra $U^c(K)$. In the last Sect. 5.4 the Lie-Butcher product on a post-Lie algebra is also shown to correspond to the natural coproduct on the dual Hopf algebra.

5.1 *Filtered Lie Algebras with Finite Dimensional Quotients*

Recall that L_p is the quotient L/L^{p+1} from Sect. 2.2. The setting in this section is k is a field of characteristic zero, and that these quotients L_p are finite dimensional as k-vector spaces. We assume that the Lie algebra L is complete with respect to this cofiltration, so we have the inverse limit

$$L = \hat{L} = \varprojlim_p L_p.$$

The dual $K^p = \mathrm{Hom}_k(L_p, k)$ is a finite dimensional Lie coalgebra. Let $K = \varinjlim_p K^p$ be the direct limit. Recall that the quotient algebra

$$U^j(L_p) = U(L_p)/F^{j+1}U(L_p).$$

The dual $U^j(L_p)^*$ is a finite dimensional coalgebra $U^c_j(K^p)$, and we have inclusions

$$\begin{array}{ccc} U^c_j(K^p) & \xrightarrow{\subseteq} & U^c_{j+1}(K^p) \\ \subseteq\downarrow & & \downarrow\subseteq \\ U^c_j(K^{p+1}) & \xrightarrow{\subseteq} & U^c_{j+1}(K^{p+1}). \end{array}$$

We have the direct limits

$$U^c(K^p) := \varinjlim_j U^c_j(K^p), \quad U^c(K) := \varinjlim_{j,p} U^c_j(K^p).$$

Lemma 5.1 *Let $T^c(K)$ be the tensor coalgebra. It is a Hopf algebra with the shuffle product. Then $U^c(K)$ is a Hopf sub-algebra of $T^c(K)$.*

Proof $U^j(L_p)$ is a quotient algebra of $T(L_p)$ and $T(L)$, and so $U^c_j(K_p)$ is a subcoalgebra of $T^c(K_p)$ and $T^c(K)$. The coproduct on $U(L_p)$, the shuffle coproduct, does not descend to a coproduct on $U^j(L_p)$. But we have a well defined map

$$U^{2j}(L_p) \to U^j(L_p) \otimes U^j(L_p)$$

compatible with the shuffle coproduct on $T(L_p)$. Dualizing this we get

$$U^c_j(K_p) \otimes U^c_j(K_p) \to U^c_{2j}(K_p)$$

and taking colimits, we get $U^c(K)$ as a subalgebra of $T^c(K)$ with respect to the shuffle product. □

Proposition 5.2 *There are isomorphisms*

a. *$L \cong Hom_k(K, k)$ of Lie algebras,*
b. *$\hat{U}(L) \cong Hom_k(U^c(K), k)$ of algebras.*
c. *The coproduct on $U^c(K)$ is dual to the completed product on $\hat{U}(L)$*

$$U^c(K) \xrightarrow{\Delta_\bullet} U^c(K) \otimes U^c(K), \quad \hat{U}(L)\hat{\otimes}\hat{U}(L) \xrightarrow{\bullet} \hat{U}(L).$$

Proof

a. Since L is the completion of the L^p, it is clear that there is a map of Lie algebras $\mathrm{Hom}_k(K, k) \to L$. We need only show that this is an isomorphism of vector spaces.

It is a general fact that for any object N in a category $\mathcal{C}$ and any indexed diagram $F : J \to \mathcal{C}$ then

$$\mathrm{Hom}(\varinjlim F(-), N) \cong \varprojlim \mathrm{Hom}(F(-), N).$$

Applying this to the category of k-vector spaces enriched in k-vector spaces (meaning that the Hom-sets are k-vector spaces), we get

$$\mathrm{Hom}_k(K, k) = \mathrm{Hom}_k(\varinjlim K^p, L) = \varprojlim \mathrm{Hom}(K^p, k) = \varprojlim L^p = \hat{L}.$$

b. This follows as in a. above.
c. This follows again by the above. Since tensor products commute with colimits we have

$$U^c(K) \otimes U^c(K) = \varinjlim_{p,j} U^c_j(K^p) \otimes U^c_j(K^p).$$

Then

$$\begin{aligned}\mathrm{Hom}_k(U^c(K) \otimes U^c(K), k) =& \mathrm{Hom}_k(\varinjlim U^c_j(K^p) \otimes U^c_j(K^p), k) \\ =& \varprojlim_{p,j} U^j(L^p) \otimes U^j(L^p) = \hat{U}(L)\hat{\otimes}\hat{U}(L).\end{aligned}$$

□

The coalgebra $U^c(K)$ is a Hopf algebra with the shuffle product. It has unit η and counit ϵ. Denote by $\star$ the convolution product on this Hopf algebra, and by $\mathbf{1}$ the identity map. Write $\mathbf{1} = \eta \circ \epsilon + J$. The Euler idempotent

$$e : U^c(K) \to U^c(K)$$

is the convolution logarithm

$$e = \log^\star(\mathbf{1}) = \log^\star(\eta \circ \epsilon + J) = J - J^{\star 2}/2 + J^{\star 3}/3 - \cdots .$$

Proposition 5.3 *The image of $U^c(K) \xrightarrow{e} U^c(K)$ is K. This inclusion of $K \subseteq U^c(K)$ is a section of the natural map $U^c(K) \to K$.*

Proof This follows the same argument as Proposition 2.1. □

This gives a map $K \to U^c(K)$. Since $U^c(K)$ is a commutative algebra under the shuffle product, we get a map from the free commutative algebra $\mathrm{Sym}(K) \to U^c(K)$.

Proposition 5.4 *This map*

$$\psi : Sym(K) \xrightarrow{\cong} U^c(K) \tag{16}$$

is an isomorphism of commutative algebras. (We later denote the shuffle product by $\sqcup\!\sqcup$*.)*

Proof By Proposition 2.2 there is an isomorphism of coalgebras

$$U(L_p) \xrightarrow{\cong} \mathrm{Sym}^c(L_p)$$

and the filtrations on these coalgebras correspond. Hence we get an isomorphism

$$U^j(L_p) \xrightarrow{\cong} \mathrm{Sym}^{c,j}(L_p).$$

Dualizing this we get

$$\mathrm{Sym}_j(K^p) \xrightarrow{\cong} U^c_j(K^p).$$

Taking the colimits of this we get the statement. □

In $\mathrm{Hom}_k(U^c(K), k)$ there are two distinguished subsets. The *characters* are the algebra homomorphisms $\mathrm{Hom}_{Alg}(U^c(K), k)$. Via the isomorphism of Proposition 5.2 they corresponds to the grouplike elements of $\hat{U}(L)$. The *infinitesimal characters* are the linear maps $\alpha : U^c(K) \to k$ such that

$$\alpha(uv) = \epsilon(u)\alpha(v) + \alpha(u)\epsilon(v).$$

We denote these as $\mathrm{Hom}_{Inf}(U^c(K), k)$.

Lemma 5.5 *Via the isomorphism in Proposition* 5.2b. *these characters correspond naturally to the following:*

a. $Hom_{Inf}(U^c(K), k) \cong Hom_k(K, k) \cong L$.
b. $Hom_{Alg}(U^c(K), k) \cong G(\hat{U}(L))$.

Proof

a. The map $U^c(K) \xrightarrow{\phi} K$ from Proposition 5.3 has kernel $k \oplus U^c(K)_+^{\sqcup\!\sqcup 2}$, by Proposition 5.4 above, where $\sqcup\!\sqcup$ denotes the shuffle product. We then see that any linear map $K \to k$ induces by composition an infinitesimal character on $U^c(K)$. Conversely given an infinitesimal character $\alpha : U^c(K) \to k$ then both k and $U^c(K)_+^{\sqcup\!\sqcup 2}$ are seen to be in the kernel, and so such a map is induced from a linear map $K \to k$ by composition with ϕ.

b. That $s : U^c(K) \to k$ is an algebra homomorphism is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} U^c(K) \otimes U^c(K) & \longrightarrow & U^c(K) \\ \downarrow{\scriptstyle s\otimes s} & & \downarrow{\scriptstyle s} \\ k \otimes_k k & \longrightarrow & k \end{array} \tag{17}$$

But this means that by the map

$$\hat{U}(L) \to \hat{U}(L)\hat{\otimes}\hat{U}(L)$$
$$s \mapsto s \otimes s.$$

Conversely given a grouplike element $s \in \hat{U}(L)$, it corresponds by Proposition 5.2b. to $s : U^c(K) \to k$, and it being grouplike means precisely that the diagram (17) commutes. □

On $\mathrm{Hom}_k(U^c(K), k)$ we also have the convolution product, which we again denote by $\star$. Note that by the isomorphism in Proposition 5.2, this corresponds to the product on $\hat{U}(L)$. Let $\mathrm{Hom}_k(U^c(K), k)_+$ consist of the α with $\alpha(1) = 0$. We then get the exponential map (we write this map without a $\star$ superscript since it is a product on the dual space)

$$\mathrm{Hom}_k(U^c(K), k)_+ \xrightarrow{\exp} \epsilon + \mathrm{Hom}_k(U^c(K), k)_+$$

given by

$$\exp(\alpha) = \epsilon + \alpha + \alpha^{\star 2}/2! + \alpha^{\star 3}/3! + \cdots .$$

This is well defined since $U^c(K)$ is a conilpotent coalgebra and $\alpha(1) = 0$. Correspondingly we get

$$\epsilon + \mathrm{Hom}_k(U^c(K), k)_+ \xrightarrow{\log} \mathrm{Hom}_k(U^c(K), k)_+$$

given by

$$\log(\epsilon + \alpha) = \alpha - \frac{\alpha^{\star 2}}{2} + \frac{\alpha^{\star 3}}{3} - \cdots .$$

Lemma 5.6 *The maps*

$$Hom_k(U^c(K), k)_+ \underset{\log}{\overset{\exp}{\rightleftarrows}} \epsilon + Hom_k(U^c(K), k)_+$$

give inverse bijections. They restrict to the inverse bijections

$$Hom_{Inf}(U^c(K), k) \underset{\log}{\overset{\exp}{\rightleftarrows}} Hom_{Alg}(U^c(K), k).$$

Proof Using the identification of Proposition 5.2 the exp and log maps above correspond to the exp and log maps in Proposition 2.10. □

Since $\mathrm{Sym}(K)$ is the free symmetric algebra on K, there is a bijection $\mathrm{Hom}_{Alg}(\mathrm{Sym}(K), k) \xrightarrow{\cong} \mathrm{Hom}_k(K, k)$. The following shows that all the various maps correspond.

Proposition 5.7 *The following diagram commutes, showing that the various horizontal bijections correspond to each other:*

$$\begin{array}{ccc}
Hom_k(K,k) & \xrightarrow{\cong} & Hom_{Alg}(Sym(K),k) \\
\| & & \uparrow \psi^* \\
Hom_k(K,k) & \xrightarrow{\exp} & Hom_{Alg}(U^c(K),k) \\
\cong \downarrow & & \downarrow \cong \\
L & \xrightarrow{\exp} & G(\hat{U}(L))
\end{array}$$

Proof That the lower diagram commutes is clear by the proof of Lemma 5.6. The middle (resp. top) map sends $K \to k$ to the unique algebra homomorphism ϕ (resp. ϕ') such that the following diagrams commute

$$\begin{array}{ccc}
K & \longrightarrow & U^c(K) \\
& \searrow & \downarrow \phi \\
& & k
\end{array}, \qquad
\begin{array}{ccc}
K & \longrightarrow & \mathrm{Sym}(K) \\
& \searrow & \downarrow \phi \\
& & k
\end{array}.$$

Since the following diagram commutes where ψ is the isomorphism of algebras

$$\begin{array}{ccc}
K & \longrightarrow & \mathrm{Sym}(K) \\
& \searrow & \downarrow \psi \\
& & U^c(K)
\end{array},$$

the commutativity of the upper diagram in the statement of the proposition follows. □

5.2 Actions of Endomorphisms

Let $E = \mathrm{End}_{Lie\ co}(K)$ be the endomorphisms of K as a Lie co-algebra, which also *respect the filtration* on K.

Proposition 5.8 *The Euler map in Proposition 5.3 is equivariant for the endomorphism action. Hence the isomorphism $\Psi : Sym(K) \to U^c(K)$ is equivariant for the action of the endomorphism group E.*

Proof The coproduct on $U^c(K)$ is clearly equivariant for E and similarly the product on $U^c(K)$ is equivariant, since $U^c(K)$ is a subalgebra of $T^c(K)$ for the shuffle product. Then if $f, g : U^c(K) \to U^c(K)$ are two equivariant maps, their convolution product $f \star g$ is also equivariant.

Since $\mathbf{1}$ and $\eta \circ \epsilon$ are equivariant for E, the difference $J = \mathbf{1} - \eta \circ \epsilon$ is so also. The Euler map $e = J - J^{\star 2}/2 + J^{\star 3}/3 - \cdots$ must then be equivariant for the action of E.

Since the image of the Euler map is K, the inclusion $K \hookrightarrow U^c(K)$ is equivariant also, and so is the map Ψ above. □

As a consequence of this the action of E on K induces an action on the dual Lie algebra L respecting its filtration. By Proposition 5.2 this again induces a diagram of actions of the following *sets*

$$\begin{array}{ccc} E \times \hat{U}(L) & \longrightarrow & \hat{U}(L) \\ \downarrow & & \downarrow \\ E \times L & \longrightarrow & L. \end{array} \tag{18}$$

5.2.1 The Free Lie Algebra

Now let $V = \oplus_{i \geq 1} V_i$ be a positively graded vector space with finite dimensional parts V_i. We consider the special case of the above that L is the completion $\widehat{\mathrm{Lie}}(V)$ of the free Lie algebra on V. Note that $\mathrm{Lie}(V)$ is a graded Lie algebra with finite dimensional graded parts. The enveloping algebra $U(\mathrm{Lie}(V))$ is the tensor algebra $T(V)$.

The graded dual vector space is $V^{\circledast} = \oplus V_i^*$ and the graded dual Lie co-algebra is $\mathrm{Lie}(V)^{\circledast}$. The Hopf algebra $U^c(\mathrm{Lie}(V)^{\circledast})$ is the shuffle Hopf algebra $T(V^{\circledast})$.

Since $\mathrm{Lie}(V)$ is the free Lie algebra on V, the endomorphisms E identifies as (note that here it is essential that we consider endomorphisms respecting the filtration)

$$\mathrm{End}_{\mathrm{Lie}\,co}(\mathrm{Lie}(V)^{\circledast}, \mathrm{Lie}(V)^{\circledast}) = \mathrm{Hom}_{\mathrm{Lie}}(\mathrm{Lie}(V), \widehat{\mathrm{Lie}}(V)). \tag{19}$$

This is a variety with coordinate ring $\mathcal{E}_V = \mathrm{Sym}(V \otimes \mathrm{Lie}(V)^{\circledast})$, which is a bialgebra. Furthermore the diagram (18) with $L = \widehat{\mathrm{Lie}}(V)$ in this case will be a morphism of varieties: Both E, L and $\hat{U}(L)$ come with filtrations and all maps are given by polynomial maps. So we get a dual diagram of coordinate rings

$$\begin{array}{ccc} \mathrm{Sym}(\mathrm{Lie}(V)^{\circledast}) & \longrightarrow & \mathcal{E}_V \otimes \mathrm{Sym}(\mathrm{Lie}(V)^{\circledast}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}(T(V)^{\circledast}) & \longrightarrow & \mathcal{E}_V \otimes \mathrm{Sym}(T(V)^{\circledast}) \end{array}.$$

But since the action of E is linear on $\widehat{\mathrm{Lie}}(V)$ and $\hat{T}(V)$, this gives a diagram

$$\begin{array}{ccc} \mathrm{Lie}(V)^{\circledast} & \longrightarrow & \mathcal{E}_V \otimes \mathrm{Lie}(V)^{\circledast} \\ \downarrow & & \downarrow \\ T^c(V^{\circledast}) & \longrightarrow & \mathcal{E}_V \otimes T^c(V^{\circledast}) \end{array},$$

and so the isomorphism $\mathrm{Sym}(\mathrm{Lie}(V)^{\circledast}) \xrightarrow{\cong} T^c(V^{\circledast})$ is an isomorphism of comodules over the algebra $\mathcal{E}_V$.

5.3 Baker-Campbell-Hausdorff on Coordinate Rings

The space K has a countable basis and so we may consider $\mathrm{Sym}(K)$ as the coordinate ring of the variety $L = \mathrm{Hom}_k(K, k)$. By the isomorphism ψ : $\mathrm{Sym}(K) \xrightarrow{\cong} U^c(K)$ of Proposition 5.4 we may think of $U^c(K)$ as this coordinate ring. Then also $U^c(K) \otimes_k U^c(K)$ is the coordinate ring of $L \times L$.

The coproduct (whose dual is the product on $\hat{U}(L)$)

$$U^c(K) \xrightarrow{\Delta_\bullet} U^c(K) \otimes_k U^c(K),$$

will then correspond to a morphism of varieties $L \times L \to L$. The following explains what it is.

Proposition 5.9 *The map $L \times L \to L$ given by*

$$(a, b) \mapsto \log^{\bullet}(\exp^{\bullet}(a) \bullet \exp^{\bullet}(b))$$

is a morphism of varieties, and on coordinate rings it corresponds to the coproduct

$$U^c(K) \xrightarrow{\Delta_\bullet} U^c(K) \otimes U^c(K).$$

This above product on L is the Baker-Campbell-Hausdorff product.

Example 5.10 Let $V = \oplus_{i \geq 1} V_i$ be a graded vector space with finite dimensional graded parts. Let $\mathrm{Lie}(V)$ be the free Lie algebra on V, which comes with a natural grading. The enveloping algebra $U(\mathrm{Lie}(V))$ is the tensor algebra $T(V)$. The dual Lie coalgebra is the graded dual $K = \mathrm{Lie}(V)^{\circledast}$, and $U^c(K)$ is the graded dual tensor coalgebra $T(V^{\circledast})$ which comes with the shuffle product. Thus the shuffle algebra $T(V^{\circledast})$ identifies as the *coordinate ring* of the Lie series, the completion $\widehat{\mathrm{Lie}}(V)$ of the free Lie algebra on V.

The coproduct on $T(V^{\circledast})$ is the deconcatenation coproduct. This can then be considered as an extremely simple codification of the Baker-Campbell-Hausdorff formula for Lie series in the completion $\widehat{\mathrm{Lie}}(V)$.

Proof If $X \to Y$ is a morphism of varieties and $A(Y) \xrightarrow{\phi} A(X)$ the corresponding homomorphism of coordinate rings, then the point p in X corresponding to the algebra homomorphism $A(X) \xrightarrow{p^*} k$ maps to the point q in Y corresponding to the algebra homomorphism $A(Y) \xrightarrow{q^*} k$ given by $q^* = \phi \circ p^*$.

Now given points a and b in $L = \mathrm{Hom}_k(K, k)$. They correspond to algebra homomorphisms from the coordinate ring $U^c(K) \xrightarrow{\tilde{a}, \tilde{b}} k$, the unique such extending a and b, and these are $\tilde{a} = \exp(a)$ and $\tilde{b} = \exp(b)$. The pair $(a, b) \in L \times L$ corresponds to the homomorphism on coordinate rings

$$\exp(a) \otimes \exp(b) : U^c(K) \otimes U^c(K) \xrightarrow{\tilde{a} \otimes \tilde{b}} k \otimes_k k = k.$$

Now via the coproduct, which is the homomorphism of coordinate rings,

$$U^c(K) \xrightarrow{\Delta_\bullet} U^c(K) \otimes U^c(K)$$

this maps to the algebra homomorphism $\exp(a) \bullet \exp(b) : U^c(K) \to k$. This is the algebra homomorphism corresponding to the following point in L:

$$\log^\bullet(\exp(a) \bullet \exp(b)) : K \to k.$$

□

5.4 Filtered Pre- and Post-Lie Algebras with Finite Dimensional Quotients

We now assume that the filtered quotients P/P^{p+1}, which again are post-Lie algebras, are all finite dimensional. Let their duals be $Q_p = \mathrm{Hom}_k(P/P^{p+1}, k)$ and $Q = \varinjlim_p Q_p$, which is a post-Lie coalgebra. We shall assume $P = \hat{P}$ is complete with respect to this filtration. Then $P = \mathrm{Hom}(Q, k)$, and $\mathrm{Sym}(Q)$ is the coordinate ring of P. There are two Lie algebra structures on P, given by $[,]$ and $[\![,]\!]$ of Definition 3.1. These correspond to the products $\bullet$ and $*$ on the enveloping algebra of P. We shall use the first product $\bullet$, giving the coproduct $\Delta_\bullet$ on $U^c(Q)$. For this coproduct Proposition 5.4 gives an isomorphism

$$\psi_\bullet : \mathrm{Sym}(Q) \xrightarrow{\cong} U^c(Q). \tag{20}$$

Due to the formula in Proposition 3.6 the product

$$P \times P \xrightarrow{\sharp} P$$

on each quotient P/P^i, is given by polynomial expressions. It thus corresponds to a homomorphism of coordinate rings

$$\mathrm{Sym}(Q) \xrightarrow{\Delta_\sharp} \mathrm{Sym}(Q) \otimes \mathrm{Sym}(Q). \tag{21}$$

Proposition 5.11 *Via the isomorphism $\psi_\bullet$ in (20) the coproduct $\Delta_\sharp$ above corresponds to the coproduct*

$$U^c(Q) \xrightarrow{\Delta_*} U^c(Q) \otimes U^c(Q),$$

which is the dual of the product $*$ on $U(P)$.

Remark 5.12 In order to identify the homomorphism of coordinate rings as the coproduct Δ_* it is essential that one uses the isomorphism $\psi_\bullet$ of (20). If one uses another isomorphism $\mathrm{Sym}(Q) \xrightarrow{\cong} U^c(Q)$ like the isomorphism ψ_* derived from the coproduct Δ_*, the statement is not correct. See also the end of the last remark below.

Remark 5.13 (The Connes-Kreimer Hopf algebra) For the free pre-Lie algebra T_C (see the next Sect. 6) this identifies the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$ as the *coordinate ring* $\mathrm{Sym}(T_C^{\circledast})$ of the Butcher series $\hat{T}_C$ under the Butcher product.

As a variety the Butcher series $\hat{T}_C$ is endowed with the Zariski topology, and the Butcher product is continuous for this topology. In [1] another finer topology on $\hat{T}_C$ is considered when the field $k = \mathbb{R}$ or $\mathbb{C}$.

Remark 5.14 (The MKW Hopf algebra) For the free post-Lie algebra P_C (see Sect. 6) it identifies the MKW Hopf algebra $T(\mathrm{OT}_C^{\circledast})$ as the *coordinate ring* $\mathrm{Sym}(\mathrm{Lie}(\mathrm{OT}_C)^{\circledast})$ of the Lie-Butcher series $\hat{P}_C = \widehat{\mathrm{Lie}}(\mathrm{OT}_C)$. A (principal) Lie-Butcher series $\ell \in \hat{P}_C$ corresponds to an element $\mathrm{Lie}(\mathrm{OT}_C)^{\circledast} \xrightarrow{\ell} k$. This lifts via the isomorphism $\psi_\bullet$ of (20) to a character of the shuffle algebra $T(\mathrm{OT}_C^{\circledast}) \xrightarrow{\tilde{\ell}} k$. That the lifting from (principal) LB series to character of the MKW Hopf algebra must be done using the inclusion $\mathrm{Lie}(\mathrm{OT}_C)^{\circledast} \hookrightarrow T(\mathrm{OT}_C^{\circledast})$ via the Euler map of

Proposition 5.3 associated to the coproduct $\Delta_\bullet$, is a technical point which has not been made explicit previously.

Proof of Proposition 5.11 Given points $a, b \in P$. They correspond to linear maps $Q \xrightarrow{a,b} k$. Via the isomorphism $\psi_\bullet$ these extend to algebra homomorphisms $U^c(Q) \xrightarrow{\tilde{a},\tilde{b}} k$, where $\tilde{a} = \exp^\bullet(a)$ and $\tilde{b} = \exp^\bullet(b)$. The pair $(a, b) \in P \times P$ then corresponds to a homomorphism of coordinate rings

$$\exp^\bullet(a) \otimes \exp^\bullet(b) : U^c(Q) \otimes U^c(Q) \xrightarrow{\tilde{a}\otimes\tilde{b}} k \otimes_k k = k.$$

Now via the coproduct associated to $*$, which is the homomorphism of coordinate rings,

$$U^c(Q) \xrightarrow{\Delta_*} U^c(Q) \otimes U^c(Q)$$

this maps to the algebra homomorphism $\exp^\bullet(a) * \exp^\bullet(b) : U^c(Q) \to k$. This is the algebra homomorphism corresponding to the following point in P:

$$\log^\bullet(\exp^\bullet(a) * \exp^\bullet(b)) : Q \to k.$$

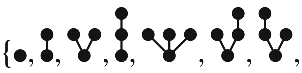
□

6 Free Pre- and Post-Lie Algebras

This section recalls free pre- and post-Lie algebras, and the notion of substitution in these algebras. We also briefly recall the notions of Butcher and Lie-Butcher series.

6.1 Free Post-Lie Algebras

We consider the set of rooted planar trees, or ordered trees:

OT = {  , ⋯ },

and let $k\mathrm{OT}$ be the k-vector space with these trees as basis. It comes with an operation $\rhd$, called *grafting*. For two trees t and s we define $t \rhd s$ to be the sum of all trees obtained by attaching the root of t with a new edge onto a vertex of s, with this new edge as the leftmost branch into the vertex of s.

If C is a set, we can color the vertices of OT with the elements of C. We then get the set OT_C of labelled planar trees. The *free post-Lie algebra* on C is the free

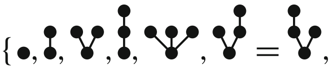
Lie algebra $P_C = \mathrm{Lie}(\mathrm{OT}_C)$ on the set of C-labelled planar trees. The grafting operation is extended to the free Lie algebra $\mathrm{Lie}(\mathrm{OT}_C)$ by using the relations from Definition 3.1. Note that P_C has a natural grading by letting $P_{C,d}$ be the subspace generated by all bracketed expressions of trees with a total number of d vertices. In particular P_C is filtered.

The enveloping algebra of P_C identifies as the tensor algebra $T(\mathrm{OT}_C)$. It was introduced and studied in [25], see also [23] for more on the computational aspect in this algebra. Its completion identifies as

$$\hat{T}(\mathrm{OT}_C) = \prod_{d \geq 0} T(\mathrm{OT}_C)_d.$$

6.2 Free Pre-Lie Algebras

Here we consider instead (non-ordered) rooted trees

$T = \{$  $, \cdots\}.$

On the vector space kT we can similarly define grafting $\rhd$. Given a set C we get the set T_C of trees labelled by C. The free pre-Lie algebra is $A_C = kT_C$, [5]. Its enveloping algebra is the symmetric algebra $\mathrm{Sym}(T_C)$, called the Grossman-Larson algebra, and comes with the ordinary symmetric product $\cdot$ and the product $*$, [26].

6.3 Butcher and Lie-Butcher Series

Recall the pre-Lie algebra $\mathcal{X}\mathbb{R}^n$ of vector fields from Example 3.2, and the corresponding power series $\mathcal{X}\mathbb{R}^n[\![h]\!]$. Let $f \in \mathcal{X}\mathbb{R}^n$ be a vector field and $A_\bullet$ the free pre-Lie algebra on one generator $\bullet$. By sending $\bullet \mapsto f$ we get a homomorphism of pre-Lie algebras $A_\bullet \to \mathcal{X}\mathbb{R}^n$ which sends a tree τ to the associated *elementary differential* f^τ, see [15, Section III.1]. If $f \in \mathcal{X}\mathbb{R}^n[\![h]\!]$ we similarly get a homomorphism of pre-Lie algebras $A_\bullet \to \mathcal{X}\mathbb{R}^n[\![h]\!]$. The natural grading on $A_\bullet$ by number of vertices of trees $|\tau|$ of a tree τ, gives a filtration and we get a map of complete pre-Lie algebras $\hat{A}_\bullet \to \mathcal{X}\mathbb{R}^n[\![h]\!]$. If we let $\bullet \to f \cdot h$ where $f \in \mathcal{X}\mathbb{R}^n$ is a vector field, then

$$\sum_{\tau \in T} \alpha(\tau)\tau \mapsto \sum_{\tau \in T} \alpha(\tau) f^\tau h^{|\tau|}$$

and the latter is called a *Butcher series*. Often this terminology is also used about the abstract form on the left side above.

In the general setting of a Lie group G. By Example 3.3, $\mathcal{X}G$ is a post-Lie algebra, and so is also the power series $\mathcal{X}G[[h]]$. Let $f \in \mathcal{X}G$ be a vector field and $P_\bullet$ the free post-Lie algebra on one generator $\bullet$. By sending $\bullet \mapsto f$ we get a homomorphism of post-Lie algebras $P_\bullet \to \mathcal{X}G$ which sends a tree τ to the associated *elementary differential* f^τ, see [18, Subsection 2.2]. We also get a map of enveloping algebras $T(OT_\bullet) \to U(\mathcal{X}G)$ which sends a forest ω to an associated differential operator f^ω. The natural grading on $P_\bullet$ by number of vertices of trees $|\tau|$ of a tree τ, gives a filtration. Sending $\bullet \mapsto f \cdot h$ we get a homomorphism of complete post-Lie algebras $\hat{P}_\bullet \to \mathcal{X}G[[h]]$. The image of an element from $\hat{P}_\bullet$ is a *Lie-Butcher series* in $\mathcal{X}G[[h]]$. Note that there is however not a really natural basis for $P_\bullet = \mathrm{Lie}(OT_\bullet)$. Therefore one usually consider instead the map from the completed enveloping algebra to the power series of differential operators ($F_\bullet$ below denotes ordered forests of ordered trees)

$$\hat{T}(OT_\bullet) \to U(\mathcal{X}G[[h]])$$
$$\sum_{\omega \in F_\bullet} \beta(\omega)\omega \mapsto \sum_{\omega \in F_\bullet} \beta(\omega) f^\omega h^{|\omega|}$$

and the latter is a *Lie-Butcher series*. The abstract form to the left is also often called a LB-series.

6.4 Substitution

In the above setting, we get by Sect. 3.2 a commutative diagram of flow maps

$$\begin{array}{ccc} \hat{P}_{\bullet,\mathrm{field}} & \xrightarrow{\Phi_P} & \hat{P}_{\bullet,\mathrm{flow}} \\ \downarrow & & \downarrow \\ \mathcal{X}G[[h]]_{\mathrm{field}} & \xrightarrow{\Phi_{\mathcal{X}G}} & \mathcal{X}G[[h]]_{\mathrm{flow}}. \end{array}$$

The field f is mapped to the flow $\Phi_{\mathcal{X}G}(f)$. By perturbing the vector field $f \to f + \delta$, it is sent to a flow $\Phi_{\mathcal{X}M}(f + \delta)$. We assume the perturbation δ is expressed in terms of the elementary differentials of f, and so it comes from a perturbation $\bullet \to \bullet + \delta' = s$. Since $\mathrm{Hom}(\bullet, P_\bullet) = \mathrm{End}_{\mathrm{postLie}}(P_\bullet)$ this gives an endomorphism of the post-Lie algebra. We are now interested in the effect of this endomorphism on the flow, called *substitution* of the perturbed vector field, and we are interested in the algebraic aspects of this action. We study this for the free post-Lie algebra P_C, but most of the discussions below are of a general nature, and applies equally well to the free pre-Lie algebra, and generalises the results of [4].

7 Action of the Endomorphism Group and Substitution in Free Post-Lie Algebras

Substitution in the free pre-Lie or free post-Lie algebras on one generator gives, by dualizing, the operation of co-substitution in their coordinate rings, which are the Connes-Kreimer and the MKW Hopf algebras. In [4] they show that co-substitution on the Connes-Kreimer algebra is governed by a bialgebra $\mathcal{H}$ such that the Connes-Kreimer algebra $\mathcal{H}_{CK}$ is a comodule bialgebra over this bialgebra $\mathcal{H}$. Moreover $\mathcal{H}_{CK}$ and $\mathcal{H}$ are isomorphic as commutative algebras. This is the notion of two bialgebras in *cointeraction*, a situation further studied in [12, 20], and [11].

In this section we do the analog for the MKW Hopf algebra, and in a more general setting, since we consider free pre- and post-Lie algebras on any finite number of generators. In this case $\mathcal{H}_{CK}$ and $\mathcal{H}$ are no longer isomorphic as commutative algebras. As we shall see the situation is understood very well by using the algebraic geometric setting and considering the MKW Hopf algebra as the coordinate ring of the free post-Lie algebra. The main results of [4] also follow, and are understood better, by the approach we develop here.

7.1 A Bialgebra of Endomorphisms

Let C be a finite dimensional vector space over the field k, and P_C the free post-Lie algebra on this vector space. It is a graded vector space $P_C = \bigoplus_{d \geq 1} P_{C,d}$ graded by the number of vertices in bracketed expressions of trees, and so has finite dimensional graded pieces. It has a graded dual

$$P_C^{\circledast} = \oplus_d \mathrm{Hom}_k(P_{C,d}, k).$$

Let $\{l\}$ be a basis for P_C. It gives a dual basis $\{l^*\}$ for $P_C^{\circledast}$. The dual of $P_C^{\circledast}$ is the completion

$$\hat{P}_C = \mathrm{Hom}_k(P_C^{\circledast}, k) = \varprojlim_d P_{C,\leq d}.$$

It is naturally a post-Lie algebra and comes with a decreasing filtration $\hat{P}_C^{d+1} = \ker(\hat{P}_C \to P_{C,\leq d})$.

Due to the freeness of P_C we have:

$$\mathrm{Hom}_k(C, P_C) = \mathrm{Hom}_{\mathrm{postLie}}(P_C, P_C) = \mathrm{End}_{\mathrm{postLie}}(P_C).$$

Denote the above vector space as E_C. If we let $\{c\}$ be a basis for C, the graded dual $E_C^{\circledast} = C \otimes_k P_C^{\circledast}$ has a basis $\{a_c(l) := c \otimes l^*\}$.

The dual of $E_C^{\circledast}$ is $\hat{E}_C = \mathrm{Hom}_k(E_C^{\circledast}, k)$ which may be written as $C^* \otimes_k \hat{P}_C$. This is an affine space with coordinate ring

$$\mathcal{E}_C := \mathrm{Sym}(E_C^{\circledast}) = \mathrm{Sym}(\mathrm{Hom}_k(C, P_C)^{\circledast}) = \mathrm{Sym}(C \otimes_k P_C^{\circledast}).$$

The filtration on $\hat{P}_C$ induces also a filtration on $\hat{E}_C$.

A map of post-Lie algebras $\phi : P_C \to \hat{P}_C$ induces a map of post-Lie algebras $\hat{\phi} : \hat{P}_C \to \hat{P}_C$. We then get the inclusion

$$\hat{E}_C = \mathrm{Hom}_{\mathrm{postLie}}(P_C, \hat{P}_C) \subseteq \mathrm{Hom}_{\mathrm{postLie}}(\hat{P}_C, \hat{P}_C).$$

If $\phi, \psi \in \hat{E}_C$, we get a composition $\psi \circ \hat{\phi}$, which we by abuse of notation write as $\psi \circ \phi$. This makes $\hat{E}_C$ into a monoid of affine varieties:

$$\hat{E}_C \times \hat{E}_C \xrightarrow{\circ} \hat{E}_C.$$

It induces a homomorphism on coordinate rings:

$$\mathcal{E}_C \xrightarrow{\Delta_\circ} \mathcal{E}_C \otimes \mathcal{E}_C.$$

This coproduct is coassociative, since $\circ$ on $\hat{E}_C$ is associative. Thus $\mathcal{E}_C$ becomes a bialgebra.

Note that when $C = \langle \bullet \rangle$ is one-dimensional, then

$$\mathcal{E}_\bullet = \mathrm{Sym}(P_\bullet^{\circledast}) \cong T^c(\mathrm{OT}_\bullet^{\circledast})$$

as *algebras*, using Proposition 5.4. The coproduct $\Delta_\circ$ considered on the shuffle algebra is, however, neither deconcatenation nor the Grossman-Larson coproduct. For the free pre-Lie algebra $A_\bullet$ instead of $P_\bullet$, a description of this coproduct is given in [4, Section 4.1/4.2].

7.1.1 Hopf Algebras of Endomorphisms

The augmentation map $P_C \to C$ gives maps

$$\mathrm{Hom}_k(C, P_C) \to \mathrm{Hom}_k(C, C)$$

and dually

$$\mathrm{Hom}_k(C, C)^{\circledast} \subseteq \mathrm{Hom}_k(C, P_C)^{\circledast} \cong C \otimes_k P_C^{\circledast}.$$

Recall that $a_c(d)$ are the basis elements of $\mathrm{Hom}_k(C, C)^{\circledast}$ (the coordinate functions on $\mathrm{Hom}_k(C, C)$), where c and d range over a basis for C. We can then invert $D =$

$\det(a_c(d))$ in the coordinate ring $\mathcal{E}_C$. This gives a Hopf algebra $\mathcal{E}_C^\times$ which is the localized ring $(\mathcal{E}_C)_D$. Another possibility is to divide $\mathcal{E}_C$ by the ideal generated by $D-1$. This gives a Hopf algebra $\mathcal{E}_C^1 = \mathcal{E}_C/(D-1)$. A third possibility is to divide $\mathcal{E}_C$ out by the ideal generated by the $a_c(d) - \delta_{c,d}$. This gives a Hopf algebra $\mathcal{E}_C^{\mathrm{Id}}$. In the case $C = \{\bullet\}$ and $P_\bullet$ is replaced with the free pre-Lie alegbra $A_\bullet$, both the latter cases give the Hopf algebra $\mathcal{H}$ in [4, Subsection 4.1/4.2].

7.2 *The Action on the Free Post-Lie Algebra*

The monoid E_C acts on P_C, and $\hat{E}_C$ acts on $\hat{P}_C$. So we get a morphism of affine varieties

$$\hat{E}_C \times \hat{P}_C \xrightarrow{\star} \hat{P}_C \tag{22}$$

called *substitution*.

Let $\mathcal{H}_C = \mathrm{Sym}(P_C^{\circledast})$ be the coordinate ring of $\hat{P}_C$. We get a homomorphism of coordinate rings called *co-substitution*

$$\mathcal{H}_C \xrightarrow{\Delta_\star} \mathcal{E}_C \otimes \mathcal{H}_C. \tag{23}$$

Note that the map in (22) is linear in the second factor so the algebra homomorphism (23) comes from a linear map

$$P_C^{\circledast} \to \mathcal{E}_C \otimes P_C^{\circledast}.$$

The action $\star$ gives a commutative diagram

$$\begin{array}{ccc} \hat{E}_C \times \hat{E}_C \times \hat{P}_C & \xrightarrow{1\times\star} & \hat{E}_C \times \hat{P}_C \\ \downarrow{\scriptstyle \circ\times\mathbf{1}} & & \downarrow{\scriptstyle\star} \\ \hat{E}_C \times \hat{P}_C & \xrightarrow{\star} & \hat{P}_C \end{array}$$

which dually gives a diagram

$$\begin{array}{ccc} \mathcal{E}_C \otimes \mathcal{E}_C \otimes \mathcal{H}_C & \longleftarrow & \mathcal{E}_C \otimes \mathcal{H}_C \\ \uparrow & & \uparrow \\ \mathcal{E}_C \otimes \mathcal{H}_C & \longleftarrow & \mathcal{H}_C. \end{array}$$

This makes $\mathcal{H}_C$ into a comodule over $\mathcal{E}_C$, in fact a comodule algebra, since all maps are homomorphisms of algebras. The Butcher product $\sharp$ on $\hat{P}_C$ is dual to the coproduct $\Delta_\star : \mathcal{H}_C \to \mathcal{H}_C \otimes \mathcal{H}_C$ by Proposition 5.11. Since $\hat{E}_C$ gives an endomorphism of post-Lie algebra we have for $a \in \hat{E}_C$ and $u, v \in \hat{P}_C$:

$$a \star (u \sharp v) = (a \star u) \sharp (a \star v).$$

In diagrams

$$\begin{array}{ccccc} \hat{E}_C \times \hat{E}_C \times \hat{P}_C \times \hat{P}_C & \xrightarrow{\mathbf{1}\times\tau\times\mathbf{1}} & \hat{E}_C \times \hat{P}_C \times \hat{E}_C \times \hat{P}_C & \xrightarrow{\star\times\star} & \hat{P}_C \times \hat{P}_C \\ {\scriptstyle \mathrm{diag}\times\mathbf{1}\times\mathbf{1}}\uparrow & & & & \downarrow{\scriptstyle \sharp} \\ \hat{E}_C \times \hat{P}_C \times \hat{P}_C & \xrightarrow{\mathbf{1}\times\sharp} & \hat{E}_C \times \hat{P}_C & \xrightarrow{\star} & \hat{P}_C \end{array}$$

which dually gives a diagram

$$\begin{array}{ccccc} \mathcal{E}_C \otimes \mathcal{E}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_C & \xleftarrow{\mathbf{1}\otimes\tau\otimes\mathbf{1}} & \mathcal{E}_C \otimes \mathcal{H}_C \otimes \mathcal{E}_C \otimes \mathcal{H}_C & \xleftarrow{\Delta_\star\otimes\Delta_\star} & \mathcal{H}_C \otimes \mathcal{H}_C \\ \downarrow & & & & \uparrow{\scriptstyle \Delta_*} \\ \mathcal{E}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_C & \xleftarrow{\mathbf{1}\otimes\Delta_*} & \mathcal{E}_C \otimes \mathcal{H}_C & \xleftarrow{\Delta_\star} & \mathcal{H}_C. \end{array}$$

This makes $\mathcal{H}_C$ into a comodule Hopf algebra over $\mathcal{E}_C$. We also have

$$a \star (u \rhd v) = (a \star u) \rhd (a \star v)$$

giving corresponding commutative diagrams, making $\mathcal{H}_C$ into a comodule algebra over $\mathcal{E}_C$.

7.2.1 The Identification with the Tensor Algebra

The tensor algebra $T(\mathrm{OT}_C)$ is the enveloping algebra of $P_C = \mathrm{Lie}(OT_C)$. The endomorphism of post-Lie co-algebras $\mathrm{End}_{\text{postLie-co}}(P_C^\circledast)$ identifies by Eq. (19) as $\hat{E}_C = \mathrm{Hom}_{\mathrm{postLie}}(C, \hat{P}_C)$. It is an endomorphism submonoid of $\mathrm{End}_{\mathrm{Lie}}(P_C^\circledast)$

By Sect. 5.2.1 the isomorphism $\mathcal{H}_C = \mathrm{Sym}(P_C^\circledast) \xrightarrow{\cong} T^c(\mathrm{OT}_C^\circledast)$ is equivariant for the action of $\hat{E}_C$ and induces a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_C & \xrightarrow{\Delta_\star} & \mathcal{E}_C \otimes \mathcal{H}_C \\ \cong\downarrow & & \downarrow\cong \\ T^c(\mathrm{OT}_C^\circledast) & \xrightarrow{\Delta_{T,\star}} & \mathcal{E}_C \otimes T^c(\mathrm{OT}_C^\circledast) \end{array} \quad (24)$$

Thus all the statements above in Sect. 7.2 may be phrased with $T^c(\mathrm{OT}_C^\circledast)$ instead of $\mathcal{H}_C$ as comodule over $\mathcal{E}_C$.

7.3 The Universal Substitution

Let K be a commutative k-algebra. We then get $P^{\circledast}_{C,K} = K \otimes_k P^{\circledast}_C$, and correspondingly we get

$$E^{\circledast}_{C,K}, \quad \mathcal{H}_{C,K} = \mathrm{Sym}(P^{\circledast}_{C,K}), \quad \mathcal{E}_{C,K} = \mathrm{Sym}(E^{\circledast}_{C,K}).$$

Let the completion $\hat{P}_{C,K} = \mathrm{Hom}(P^{\circledast}_{C,K}, K)$. (Note that this is not $K \otimes_k \hat{P}_C$ but rather larger than this.) Similarly we get $\hat{E}_{C,K}$. The homomorphism of coordinate rings $\mathcal{H}_{C,K} \to \mathcal{E}_{C,K} \otimes_K \mathcal{H}_{C,K}$ corresponds to a map of affine K-varieties (see Remark 4.2)

$$\hat{E}_{C,K} \times \hat{P}_{C,K} \to \hat{P}_{C,K}. \tag{25}$$

A K-point A in the affine variety $\hat{E}_{C,K}$ then corresponds to an algebra homomorphism $\mathcal{E}_{C,K} \xrightarrow{A^*} K$, and K-points $p \in \hat{P}_{C,K}$ corresponds to algebra homomorphisms $\mathcal{H}_{C,K} \xrightarrow{p^*} K$.

In particular the map obtained from (25), using $A \in \hat{E}_{C,K}$:

$$\hat{P}_{C,K} \xrightarrow{A_\star} \hat{P}_{C,K} \tag{26}$$

corresponds to the morphism on coordinate rings

$$\mathcal{H}_{C,K} \to \mathcal{H}_{C,K} \otimes_K \mathcal{E}_{C,K} \xrightarrow{\mathbf{1}\otimes A^*} \mathcal{H}_{C,K} \otimes_K K = \mathcal{H}_{C,K} \tag{27}$$

which due to (26) being linear, comes from a K-linear map

$$P^{\circledast}_{C,K} \to P^{\circledast}_{C,K}.$$

Now we let K be the commutative algebra $\mathcal{E}_C = \mathrm{Sym}(E^{\circledast}_C)$. Then

$$\mathcal{E}_{C,K} = K \otimes_k \mathrm{Sym}(E^{\circledast}_C) = \mathrm{Sym}(E^{\circledast}_C) \otimes \mathrm{Sym}(E^{\circledast}_C).$$

There is a canonical algebra homomorphism

$$\mathcal{E}_{C,K} \xrightarrow{\mu} K \tag{28}$$

which is simply the product

$$\mathrm{Sym}(E^{\circledast}_C) \otimes_k \mathrm{Sym}(E^{\circledast}_C) \xrightarrow{\mu} \mathrm{Sym}(E^{\circledast}_C).$$

Definition 7.1 Corresponding to the algebra homomorphism μ of (28) is the point U in $\hat{E}_{C,K} = \mathrm{Hom}_k(C_K, \hat{P}_{C,K})$. This is the *universal* map (here we use the completed tensor product):

$$C \to C \otimes (P_C^{\circledast} \hat{\otimes} P_C) \tag{29}$$

sending

$$c \mapsto c \otimes \sum_{\substack{l \text{ basis} \\ \text{element of } P_C}} l^* \otimes l = \sum_l a_c(l) \otimes l$$

Using this, (26) becomes the *universal* substitution, the K-linear map

$$\hat{P}_{C,K} \xrightarrow{U_\star} \hat{P}_{C,K}.$$

Let $H = \mathrm{Hom}(C, P_C)^{\circledast}$, the degree one part of $K = \mathcal{E}_C$, and $P_{C,H} = H \otimes_k P_C$. Note that the universal map (29) is a map from C to $\hat{P}_{C,H}$.

If $a \in \hat{E}_C$ is a specific endomorphism, it corresponds to an algebra homomorphism (character)

$$K = \mathcal{E}_C \xrightarrow{\alpha} k$$
$$a_c(l) = c \otimes l^* \mapsto \alpha(c \otimes l^*).$$

Then $U_\star$ induces the substitution $\hat{P}_C \xrightarrow{a\star} \hat{P}_C$ by sending each coefficient $a_c(l) \in K$ to $\alpha(c \otimes l^*) \in k$.

The co-substitution $\mathcal{H}_C \xrightarrow{\Delta_\star} \mathcal{E}_C \otimes \mathcal{H}_C$ of (23) induces a homomorphism

$$\mathcal{E}_C \otimes \mathcal{H}_C \to \mathcal{E}_C \otimes \mathcal{E}_C \otimes \mathcal{H}_C \to \mathcal{E}_C \otimes \mathcal{H}_C$$

which is seen to coincide with the homomorphism (27) when $K = \mathcal{E}_C$. The universal substitution therefore corresponds to the map on coordinate rings which is the co-substitution map, suitably lifted.

Recall that the tensor algebra $T(\mathrm{OT}_C)$ identifies as the forests of ordered trees OF_C. We may then write $T^c(\mathrm{OT}_C^{\circledast}) = \mathrm{OF}_C^{\circledast}$. By the diagram (24) the co-substitution $\mathcal{H}_{C,K} \xrightarrow{\Delta_\star} \mathcal{H}_{C,K}$ identifies as a map $\mathrm{OF}_{C,K}^{\circledast} \xrightarrow{U_\star^T} \mathrm{OF}_{C,K}^{\circledast}$ and we get a commutative diagram and its dual

$$\begin{array}{ccc} \mathrm{OF}^{\circledast}_{C,K} & \xrightarrow{U^T_\star} & \mathrm{OF}^{\circledast}_{C,K} \\ \downarrow & & \downarrow \\ P^{\circledast}_{C,K} & \xrightarrow{U_\star} & P^{\circledast}_{C,K} \end{array}, \qquad \begin{array}{ccc} \hat{P}_{C,K} & \xrightarrow{U_\star} & \hat{P}_{C,K} \\ \downarrow & & \downarrow \\ \hat{\mathrm{OF}}_{C,K} & \xrightarrow{U_\star} & \hat{\mathrm{OF}}_{C,K} \end{array}.$$

We may restrict this to ordered trees and get

$$\mathrm{OF}^{\circledast}_{C,K} \xrightarrow{\overline{U}^T_\star} \mathrm{OT}^{\circledast}_{C,K}, \quad \hat{\mathrm{OT}}_{C,K} \xrightarrow{\overline{U}_\star} \hat{\mathrm{OF}}_{C,K}.$$

We may also restrict and get

$$C_K \to \hat{P}_{C,K} \to \hat{\mathrm{OF}}_{C,K},$$

with dual map

$$U^t : \mathrm{OF}^{\circledast}_{C,K} \to P^{\circledast}_{C,K} \to C^*_K \tag{30}$$

For use in Sect. 7.4.1, note that (29) sends C to $\hat{P}_{C,H}$ where $H = \mathrm{Hom}_k(C, P_C)^{\circledast} \subseteq K$ is the graded dual of E_C. A consequence is that $\mathrm{OF}^{\circledast}_C \subseteq \mathrm{OF}^{\circledast}_{C,K}$ is mapped to $C^*_H \subseteq C^*_K$ by U^t.

7.4 Recursion Formula

The universal substitution is described in [18], and we recall it. By attaching the trees in a forest to a root $c \in C$, there is a natural isomorphism

$$\mathrm{OT}_C \cong \mathrm{OF}_C \otimes C$$

and dually

$$\mathrm{OF}^{\circledast}_C \otimes C^* \xrightarrow{\cong} \mathrm{OT}^{\circledast}_C \tag{31}$$

Here we denote the image of $\omega \otimes \rho$ as $\omega \curvearrowright \rho$.

Proposition 7.2 ([18]) *The following gives a partial recursion formula for $\overline{U}^T_\star$, the universal co-substitution followed by the projection onto the dual ordered trees:*

$$\overline{U}^T_\star(\omega) = \sum_{\Delta_\triangleright(\omega)} U^T_\star(\omega^{(1)}) \curvearrowright U^t(w^{(2)}).$$

Proof Recall the following general fact. Two maps $V \xrightarrow{\phi} W$ and $W^* \xrightarrow{\psi} V^*$ are dual iff for all $v \in V$ and $w^* \in W^*$ the pairings

$$\langle v, \psi(w^*)\rangle = \langle \phi(v), w^*\rangle.$$

We apply this to $\phi = \overline{U_\star}$ and

$$\psi : \mathrm{OF}^{\circledast}_{C,K} \xrightarrow{\Delta_\triangleright} \mathrm{OF}^{\circledast}_{C,K} \otimes \mathrm{OF}^{\circledast}_{C,K} \xrightarrow{U^T_\star \otimes U^t} \mathrm{OF}^{\circledast}_{C,K} \otimes C^*_K \xrightarrow{\curvearrowright} \mathrm{OT}^{\circledast}_{C,K}.$$

We must then show that

$$\sum_{\Delta_\triangleright(\omega)} \langle t, U^T_\star(\omega^{(1)}) \curvearrowright U^t(w^{(2)})\rangle = \langle \overline{U}_\star(t), \omega\rangle$$

So let $t = f \triangleright c$. Using first the above fact on the map (31) and its dual:

$$\begin{aligned}
\sum_{\Delta_\triangleright(\omega)} \langle t, U^T_\star(\omega^{(1)}) \curvearrowright U^t(w^{(2)})\rangle &= \sum_{\Delta_\triangleright(\omega)} \langle f \otimes c, U^T_\star(\omega^{(1)}) \otimes U^t(\omega^{(2)})\rangle \\
&= \sum_{\Delta_\triangleright(\omega)} \langle f, U^T_\star(\omega^{(1)})\rangle \cdot \langle c, U^t(\omega^{(2)})\rangle \\
&= \sum_{\Delta_\triangleright(\omega)} \langle U_\star(f), \omega^{(1)}\rangle \cdot \langle U(c), \omega^{(2)}\rangle \\
&= \langle U_\star(f) \otimes U(c), \Delta_\triangleright(\omega)\rangle \\
&= \langle U_\star(f) \triangleright U(c), \omega\rangle \\
&= \langle \overline{U}_\star(f \triangleright c), \omega\rangle = \langle \overline{U}_\star(t), \omega\rangle
\end{aligned}$$

□

We now get the general recursion formula, Theorem 3.7, in [18].

Proposition 7.3

$$U^T_\star(\omega) = \sum_{\Delta_\bullet(\omega)} U^T_\star(\omega_1) \cdot \overline{U}^T_\star(\omega_2).$$

Proof Given a forest $f \cdot t$ where t is a tree. We will show

$$\langle U_\star(ft), \omega\rangle = \sum_{\Delta_\bullet(\omega)} \langle ft, U^T_\star(\omega_1) \cdot \overline{U}^T_\star(\omega_2)\rangle.$$

We have:

$$\langle U_\star(ft), \omega\rangle = \langle U_\star(f) \cdot U_\star(t), \omega\rangle.$$

Since concatenation and deconcatenation are dual maps, this is

$$\begin{aligned}
&= \sum_{\Delta_\bullet(\omega)} \langle U_\star(f) \otimes U_\star(t), \omega_1 \otimes \omega_2\rangle \\
&= \sum_{\Delta_\bullet(\omega)} \langle U_\star(f), \omega_1\rangle \cdot \langle U_\star(t), \omega_2\rangle \\
&= \sum_{\Delta_\bullet(\omega)} \langle f, U_\star^T(\omega_1)\rangle \cdot \langle t, \overline{U}_\star^T(\omega_2)\rangle.
\end{aligned}$$

Since $\overline{U}_\star^T(\omega_2)$ is a dual tree, this is:

$$= \sum_{\Delta_\bullet(\omega)} \langle ft, U_\star^T(\omega_1) \cdot \overline{U}_\star^T(\omega_2)\rangle.$$

□

7.4.1 The Case of One Free Generator

Now consider the case that $C = \langle \bullet \rangle$ is a one-dimensional vector space. Recall the isomorphism $\psi : \mathcal{E}_\bullet \cong T(\mathrm{OT}_\bullet^\circledast)$ as algebras but the coproduct on this is different from $\mathcal{H}_\bullet \cong T(\mathrm{OT}_\bullet^\circledast)$. To signify the difference, we denote the former by $T^\circ(\mathrm{OT}_\bullet^\circledast)$. It is the free algebra on the alphabet $a_\bullet(t)$ where the t are ordered trees. Multiplication on $\mathcal{E}_\bullet = Sym(P_\bullet^\circledast)$ corresponds to the shuffle product on $T^\circ(\mathrm{OT}_\bullet^\circledast)$.

The coproduct

$$\mathcal{H}_\bullet \xrightarrow{\Delta_\star} \mathcal{E}_\bullet \otimes_k \mathcal{H}_\bullet$$

may then by Sect. 7.2.1 be written as

$$T(\mathrm{OT}_\bullet^\circledast) \xrightarrow{\Delta_\star} T^\circ(\mathrm{OT}_\bullet^\circledast) \otimes_k T(\mathrm{OT}_\bullet^\circledast) = K \otimes_k T(\mathrm{OT}_\bullet^\circledast).$$

The two bialgebras $T(\mathrm{OT}_\bullet^\circledast)$ and $T^\circ(\mathrm{OT}_\bullet^\circledast)$ are said to be in cointeraction, a notion studied in [4, 12, 20], and [11].

The element $U^t(\omega^{(2)})$ is in $C_K^* \cong K$. By the comment following (30) it is in

$$C_H^* = \mathrm{Hom}_k(\bullet, P_\bullet)^\circledast \otimes_k \bullet^* \cong P_\bullet^\circledast.$$

Then $U^t(\omega^{(2)})$ is simply the image of $\omega^{(2)}$ by the natural projection $T(\mathrm{OT}^{\circledast}_{\bullet}) \to P^{\circledast}_{\bullet}$. We may consider $U^t(\omega^{(2)})$ as an element of $K \cong T^{\circ}(\mathrm{OT}^{\circledast}_{\bullet})$ via the isomorphism ψ above. We are then using the Euler idempotent map

$$T(\mathrm{OT}^{\circledast}_{\bullet}) \xrightarrow{\pi} T(\mathrm{OT}^{\circledast}_{\bullet}) \cong T^{\circ}(\mathrm{OT}^{\circledast}_{\bullet}),$$

so that $U^t(\omega^{(2)}) = \pi(\omega^{(2)})$.

Let B_+ be the operation of attaching a root to a forest in order to make it a tree. By a decorated shuffle $\sqcup\!\sqcup$ below we mean taking the shuffle product of the corresponding factors in $K = T^{\circ}(\mathrm{OT}^{\circledast}_{\bullet})$. By the decorated $\cdot$ product we mean concatenating the corresponding factors in $T(\mathrm{OT}^{\circledast}_{\bullet})$. Then we may write the recursion of Proposition 7.3 as:

Proposition 7.4

$$\begin{aligned}\Delta_{\star}(\omega) &= \sqcup\!\sqcup_{13} \cdot_{24} \Delta_{\star}(\omega_1) \otimes \overline{U}^T_{\star}(\omega_2)\\ &= \sqcup\!\sqcup_{135} \cdot_{24} \Delta_{\star}(\omega_1) \otimes B^{+}(\Delta_{\star}(\omega_2^{(1)})) \otimes \pi(\omega_2^{(2)})\end{aligned}$$

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