Numerical Integration of Differential Equations on Homogeneous Manifolds *

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Abstract. We present an overview of intrinsic integration schemes for differential equations evolving on manifolds, paying particular attention to homogeneous spaces. Various examples of applications are introduced, showing the generality of the methods. Finally we discuss abstract Runge–Kutta methods. We argue that homogeneous spaces are the natural structures for the study and the analysis of these methods.

1 Introduction

In the last few years there has been an increasing interest in the design of numerical integration techniques which preserve important qualitative properties of differential equations. This includes the recent work on preserving the symplectic structure of Hamiltonian systems [15], orthogonality [5], isospectrality [2] and on designing numerical methods which stay on a prescribed manifold [3, 4, 6, 7, 11, 12, 13].

In this paper we are concerned with the latter problem, in particular numerical integration of differential equations evolving on homogeneous spaces. The structure of a homogeneous space is both specific enough to allow the definition and the analysis of numerical integration methods, and, at the same time, general enough to include most domains in mathematical modeling. Some examples are:

- Any Lie group; e.g. $\mathrm{SO}(n)$ associated with orthogonal flows and $\mathrm{SL}(n)$ related to volume preserving flows.
- The classical manifolds: $\mathbb{R}^n$, the n-sphere S^n, projective spaces $\mathrm{RP}(n)$, the Stiefel and Grassman manifolds.
- Any group of diffeomorphisms from a manifold $\mathcal{M}$ onto itself, acting transitively on $\mathcal{M}$.

There are two main approaches for integrating differential equations on manifolds, associated with *embedded* and *intrinsic* methods. The first consists of methods where the manifold is embedded in a vector space and a classical integration scheme is employed. The work of Calvo, Iserles and Zanna [3, 7] shows

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that except for a few special, yet important cases, it is impossible to devise classical integration schemes such that the numerical solution will stay on the correct manifold. Intrinsic methods are based on natural movements on the manifold, therefore they are independent of the particular embedding and remain on the correct manifold by construction. The main price we pay for this is that we need to compute exponential mappings, either as matrix exponentials or by computing flows of special vectorfields.

In this paper we present a brief overview of the intrinsic methods introduced in [4, 6, 13, 17]. We will focus on the main structures and ideas rather than the technical details, and present several applications.

Finally we will address the question of abstract Runge–Kutta (RK) methods; what are the basic operations needed to formulate RK methods in an abstract setting? This is an important topic in object-oriented program design and a fundamental issue in understanding the foundations of the methods themselves. We will argue that homogeneous spaces are the natural structure to formulate RK methods, and unless we have this structure, it seems difficult to define and analyze RK-type integration schemes.

2 Numerical Integration on Lie Groups

A Lie group is defined as a manifold G equipped with a binary operation $\cdot$: $\mathrm{G}\times\mathrm{G} \rightarrow \mathrm{G}$ satisfying the axioms of a group, such that the mappings $(a, b) \mapsto a\cdot b$ and $a \mapsto a^{-1}$ are smooth. We define right multiplications in the group as:

$$R_a(b) = b\cdot a, \qquad \text{for } a, b \in \mathrm{G}.$$

Examples of Lie groups are matrix Lie groups, where G consist of a set of non-singular matrices and the product is the matrix product. We let $\mathrm{GL}(n)$ denote the set of all non-singular $n\times n$ matrices. Another example is given by $\mathrm{Diff}(\mathcal{M})$, the set of diffeomorphisms on a manifold $\mathcal{M}$, where the product is the composition of diffeomorphisms.

The Lie algebra of the group G is defined as its tangent space at the identity, $\mathfrak{g} = T\mathrm{G}|_e$, and it has the structure of a (real or complex) linear space equipped with a bilinear skew-symmetric form $[\_,\_] : \mathfrak{g}\times\mathfrak{g} \rightarrow \mathfrak{g}$, called the *Lie bracket.* If $\mathrm{G} = \mathrm{GL}(n)$ then $\mathfrak{g}$ is the space of $n \times n$ matrices and the Lie bracket is the matrix commutator $[u, v] = uv - vu$. If $\mathrm{G} = \mathrm{Diff}(\mathcal{M})$, then $\mathfrak{g} = \mathfrak{X}(\mathcal{M})$ is the set of tangent vectorfields on $\mathcal{M}$ and the Lie bracket is given as the Lie–Jacobi bracket of vectorfields, which, given $F, G \in \mathfrak{X}(\mathcal{M})$, yields

$$[F, G]^i = \sum_j \left(F^j \frac{\partial G^i}{\partial x^j} - G^j \frac{\partial F^i}{\partial x^j} \right).$$

The exponential function is a mapping $\exp : \mathfrak{g} \rightarrow \mathrm{G}$, such that $t \mapsto \exp(tv)$ is a one-parameter subgroup of G. In the case of matrix Lie groups, it reduces to the matrix exponential

$$\exp(tv) = \sum_{j=0}^{\infty} \frac{(tv)^j}{j!},$$

while on Diff($\mathcal{M}$) it returns the flow of a given vector field. This can be expressed as

$$\frac{\partial}{\partial t} \exp(tF)(y)\bigg|_{t=0} = F(y).$$

2.1 The Crouch–Grossman (CG) Methods

The starting point for the methods of Crouch and Grossman, as presented in [4, 11], is the introduction of a *frame* on the manifold $\mathcal{M}$, i.e. a set of smooth vectorfields $\{E_1, \ldots, E_d\} \subset \mathfrak{X}(\mathcal{M})$, which at each point $p \in \mathcal{M}$ span the tangent space $T\mathcal{M}|_p$. It follows that an ordinary differential equation on $\mathcal{M}$ can be written as

$$y' = F(y) = \sum_{i=1}^{d} E_i(y) f_i(y),$$

where $f_i : \mathcal{M} \to \mathbb{R}$ are smooth scalar functions. We denote by F_p the vectorfield F with coefficients frozen at p relative to the frame:

$$F_p(y) = \sum_{i=1}^{d} E_i(y) f_i(p). \tag{1}$$

The explicit s-stage Crouch–Grossman method with stepsize h is given as:

Algorithm 1 (RK–CG):

$p = y_k$
$\nu_1 = p$
`for` $r = 2, \ldots, s$
  $\nu_r = \exp(ha_{r,r-1}F_{\nu_{r-1}}) \circ \exp(ha_{r,r-2}F_{\nu_{r-2}}) \circ \cdots \circ \exp(ha_{r,1}F_{\nu_1})(p)$
`end`
$y_{k+1} = \exp(hb_s F_{\nu_s}) \circ \exp(hb_{s-1} F_{\nu_{s-1}}) \circ \cdots \circ \exp(hb_1 F_{\nu_1})(p)$

where the exponentiation of a vectorfield denotes its flow.

If $\mathcal{M} = \mathrm{G}$ is a matrix Lie group and $E_i(y) = v_i y$ are right invariant vectorfields generated by a basis $\{v_1, \ldots, v_d\}$ spanning $\mathfrak{g}$, we can compute the flows via the matrix exponential

$$\exp\left(\sum_{i=1}^{d} \alpha_i E_i\right)(y) = \exp\left(\sum_{i=1}^{d} \alpha_i v_i\right) y.$$

Crouch and Grossman construct methods of order three. Owren and Marthinsen [11] develop a systematic order theory for this type of methods. They present a fourth-order method and prove that five stages are required to obtain order four. Due to non-commutative effects, the number of order conditions for this class of methods grows faster than for classical Runge–Kutta schemes. It is therefore difficult to construct higher order methods of this type, and it is at the moment unknown if methods of order five exist.

2.2 The Munthe-Kaas (MK) Methods

The Runge–Kutta-type methods of Munthe-Kaas were introduced in [12] and further developed in [13]. The main difference between the CG and MK methods is that whereas CG perform the approximations in the Lie group, MK does them in the Lie algebra. In other words, CG advance by a composed product of exponentials, while MK first combines elements in the Lie algebra, and then advances by a single exponential mapping. It turns out that the order theory for the MK approach is much simpler than for the CG approach due to the fact that, whereas operations in the Lie group are nonlinear, the Lie algebra is a linear space.

A general initial value problem on a Lie group can be written as

$$y_t' = R_{y_t}'(f(y_t)), \qquad y_0 = p, \tag{2}$$

where $f : \mathrm{G} \to \mathfrak{g}$. If G is a matrix Lie group, (2) becomes

$$y_t' = f(y_t)y_t, \qquad y_0 = p. \tag{3}$$

The general s-stage MK methods with stepsize h for the initaial value problems (2),(3) is:

Algorithm 2 (RK–MK):

$y_0 = p$
`for` $n = 0, 1, \ldots$
    `for` $i = 1, 2, \ldots, s$
        $u_i = h \sum_{j=1}^{s} a_{i,j} k_j$
        $\tilde{u}_i = \zeta_i(u_i, k_1, k_2, \ldots, k_s)$
        $k_i = f(\exp(\tilde{u}_i) \cdot y_n)$
    `end`
    $v = h \sum_{j=1}^{s} b_j k_j$
    $\tilde{v} = \zeta(v, k_1, k_2, \ldots, k_s)$
    $y_{n+1} = \exp(\tilde{v}) \cdot y_n$
`end`

where $u_i, \tilde{u}_i, k_i, v, \tilde{v} \in \mathfrak{g}$ *and* $y_i \in \mathrm{G}$. *The real constants* $a_{i,j}, b_j$ *and the correction functions* $\zeta_i()$ *and* $\zeta()$ *determine a particular scheme.*

In [13] the order theory for such methods is developed in terms of a *Lie–Butcher* series for the numerical and the analytical solution. The starting point of the analysis is the identification of the tangent space $T\mathrm{G}|_p$ at an arbitrary point p with $\mathfrak{g}$ by means of the right multiplication. (In other words, $T\mathrm{G}$ and $\mathfrak{g}$ are related by the canonical right-invariant Maurer–Cartan form on G). In this respect, any curve on G may be lifted to a curve on $\mathfrak{g}$, and the order conditions arise by comparing terms in the Lie–Butcher series for the analytical and the numerical solution on $\mathfrak{g}$. The main result of [13] is the following:

Theorem 1. *If the coefficients $a_{i,j}$ and b_j satisfy the classical Runge–Kutta order conditions up to order q, and the correction functions $\zeta_i()$ and $\zeta()$ satisfy certain conditions given in [13], then Algorithm 2 defines a q'th order method on an arbitrary Lie group.*

An important consequence of the above result is that the task of computing the correction functions is independent of the task of finding the coefficients $a_{i,j}$ and b_j, thus when the correction functions are constructed to order q, any classical RK method of the same order can be turned into a MK-type method of order q on an arbitrary Lie group. In Algorithm 3, presented in Sect. 3, we give the correction functions for order four.

2.3 The Method of Iterated Commutators

The method of iterated commutators, introduced by Iserles in [6], is given a lengthy treatment in another paper in this volume [18] as well as in [17]. We will therefore not repeat the algorithm here, but just state its basic properties. In its original form it is presented as an algorithm for solving linear matrix differential equations with variable coefficients. It is naturally generalized to Lie-group differential equations of the form

$$y_t' = R_{y_t}'(f(t)), \qquad y_0 = p,$$

where $f : \mathbb{R} \to \mathfrak{g}$ depends on time but not on space. Such equations are called *equations of Lie type*. The method advances by a product of exponential mappings, and in that respect it somehow resembles the RK–CG algorithm. Its theoretical explanation is, however, very different from both RK–CG and RK–MK schemes. The method can be explained as a numerical implementation of the technique of Lie reduction. In the case where G is solvable, the Lie reduction technique can be used to express the solution in terms of quadratures. The numerical technique is based on numerical quadratures and it works also for non-solvable groups. Zanna [17] has recently generalized this approach to the general case when $f : \mathrm{G} \to \mathcal{M}$.

3 Homogeneous Spaces

A *left action* of a Lie group G on a manifold $\mathcal{M}$ is a smooth map $\lambda : \mathrm{G} \times \mathcal{M} \to \mathcal{M}$ such that

$$\lambda(e, m) = m,$$

$$\lambda(g_1 \cdot g_2, m) = \lambda(g_1, \lambda(g_2, m)), \qquad g_1, g_2 \in \mathrm{G}, \quad m \in \mathcal{M},$$

where e denotes the identity element in G. We henceforth assume that the action is *transitive*, which means that for any two points $m_1, m_2 \in \mathcal{M}$ there exist a $g \in \mathrm{G}$ such that $\lambda(g, m_1) = m_2$. A manifold with a transitive Lie group action is called a *homogeneous space*.

For a given fixed point $p \in \mathcal{M}$, let G_p denote the *stabilizer* of p,

$$\mathrm{G}_p = \{\, g \in \mathrm{G} \mid \lambda(g, p) = p \,\}.$$

The set G_p is a subgroup of G. A fundamental result [1] states that G/G_p and $\mathcal{M}$ are naturally diffeomorphic, where G/G_p denotes the left cosets of G_p in G. Hence, we may equivalently define a homogeneous space as the quotient of two Lie groups.

As for Lie groups we used the right multiplication to identify $\mathfrak{g}$ with $T\mathrm{G}|_p$, we now use λ to map $\mathfrak{g}$ onto $T\mathcal{M}|_p$. Let $\lambda_p : \mathrm{G} \to \mathcal{M}$ be defined as $\lambda_p(g) = \lambda(g, p)$. Then,

$$\lambda_p{}' : \mathfrak{g} \to T\mathcal{M}|_p\,, \qquad v \mapsto \lambda_p{}'(v) = \left.\frac{\partial}{\partial t}\lambda(\exp(tv), p)\right|_{t=0}. \tag{4}$$

This mapping is surjective since λ is transitive. It is, however, not injective. If $\mathfrak{g}_p \subset \mathfrak{g}$ denotes the Lie algebra of G_p one may verify that $T\mathcal{M}|_p \simeq \mathfrak{g}/\mathfrak{g}_p$. Thus two vectors $v, w \in \mathfrak{g}$ correspond to the same direction in $T\mathcal{M}|_p$ if and only if $v - w \in \mathfrak{g}_p$.

We now have all the necessary tools to define the order for methods on $\mathcal{M}$, and to derive the order conditions. It is a tedious work, yet straightforward, to check that all definitions and results which in [13] are stated for integration on G, do hold for integration on $\mathcal{M}$. The correspondence is obtained by replacing the right multiplication on G, $R_p(g) = g \cdot p$, where both $g, p \in \mathrm{G}$, with the action $\lambda_p(g) = \lambda(g, p)$, where $g \in \mathrm{G}, p \in \mathcal{M}$. The general initial value problem on $\mathcal{M}$ can be written as:

$$y_t' = \lambda_{y_t}{}'(f(y_t)), \qquad y_0 = p, \tag{5}$$

where $f : \mathcal{M} \to \mathfrak{g}$.

Algorithm 3 is a fourth-order integration scheme for such equations. The correction functions ζ_i and ζ are given for order four.

Algorithm 3 Explicit fourth-order RK–MK on homogeneous spaces: *Choose the coefficients $a_{i,j}$ and b_j of a classical s-stage, fourth-order explicit RK method. Let $c_i = \sum_{j=1}^{s} a_{i,j}$, and $d_i = \sum_{j=1}^{s} a_{i,j} c_j$. Compute the coefficients (m_1, m_2, m_3) by solving the linear system:*

$$(m_1 \;\; m_2 \;\; m_3) \begin{pmatrix} c_2 & c_2^2 & 2d_2 \\ c_3 & c_3^2 & 2d_3 \\ c_4 & c_4^2 & 2d_4 \end{pmatrix} = (1 \;\; 0 \;\; 0)\,.$$

Then the fourth-order RK–MK algorithm for (5) is given by

```
y_0 = p
for n = 0, 1, 2, ...
    I_1 = k_1 = f(y_n)
    for i = 2, ..., s
        u_i = h Σ_{j=1}^{i-1} a_{i,j} k_j
        ũ_i = u_i - (c_i h / 6)[I_1, u_i]
        k_i = f(λ(exp(ũ_i), y_n))
    end
    I_2 = (m_1(k_2 - I_1) + m_2(k_3 - I_1) + m_3(k_4 - I_1)) / h
    v = h Σ_{j=1}^{s} b_j k_j
    ṽ = v - (h/4)[I_1, v] - (h^2/24)[I_2, v]
    y_{n+1} = λ(exp(ṽ), y_n)
end
```

We can for instance base the algorithm on the classical four-stage fourth-order RK scheme whose coefficients are

$$\begin{array}{l} a_{2,1} = \frac{1}{2},\ a_{3,2} = \frac{1}{2},\ a_{4,3} = 1, \text{ all other } a_{i,j} = 0, \\ \ b_1 = \frac{1}{6}, \quad b_2 = \frac{1}{3}, \quad b_3 = \frac{1}{3},\ b_4 = \frac{1}{6}, \end{array}$$

to find

$$\begin{array}{l} c_1 = 0, \ \ c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2}, \quad c_4 = 1, \\ d_1 = 0, \ \ d_2 = 0, \quad d_3 = \frac{1}{4}, \quad d_4 = \frac{1}{2}, \\ m_1 = 2,\ m_2 = 2,\ m_3 = -1. \end{array}$$

The RK–MK algorithms are abstractly formulated, hence admit several different interpretations. Some of them will be presented in the next section.

4 Examples

4.1 The Classical Case $\mathbb{R}^n$

The real vector space $\mathbb{R}^n$ is a Lie group where the group product is the sum of vectors. The Lie algebra is also $\mathbb{R}^n$, the Lie bracket is given as $[u, v] = 0$ and the exponential as $\exp(u) = u$. The initial value problem (2) becomes

$$y_t' = f(y_t), \qquad \text{for } f : \mathbb{R}^n \to \mathbb{R}^n.$$

We may verify that the RK–CG and RK–MK schemes both reduce to their classical RK counterpart and that the method of iterated commutators in this case becomes solution of $y' = f(t)$ by a single quadrature.

4.2 The n-Sphere

Let $\mathrm{G} = \mathrm{SO}(n+1)$ be the Lie group of real orthogonal $(n+1)\times(n+1)$ matrices with determinant equal to one. Its Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ is the set of skew-symmetric matrices. The n-sphere $\mathcal{M} = S^n = \{\, y \in \mathbb{R}^{n+1} \mid y^T y = 1 \,\}$ is the homogeneous space related to the action given by the matrix-vector product $\lambda(A, y) = Ay$. Thus, $\lambda_y{}'(V) = Vy$, and the general initial value problem on the sphere is

$$y_t' = f(y_t)y_t, \qquad y_0 = p,$$

where $f : S^n \to \mathfrak{so}(n+1)$.

4.3 Isospectral Flows

This topic is covered thoroughly in [2]. Let m_0 be a symmetric $n \times n$ matrix[3], $m_0 \in \mathrm{Sym}(n)$. The matrix differential equation

$$y' = [f(y), y] = f(y)y - yf(y), \qquad y_0 = m_0, \tag{6}$$

where $f(y) \in \mathfrak{so}(n)$, is an example of an isospectral flow evolving in $\mathrm{Sym}(n)$. The adjective 'isospectral' reflects the fact that the eigenvalues of y are invariant under the flow. We define the following submanifold of $\mathrm{Sym}(n)$, $\mathcal{M} = \{\, y = am_0a^T \mid a \in \mathrm{SO}(n) \,\}$, and define the action $\lambda : \mathrm{SO}(n)\times\mathcal{M} \to \mathcal{M}$ as

$$\lambda(a, y) = aya^T.$$

The action λ is transitive. Furthermore, from (4), using the fact that in $a^T = a^{-1}$, we get

$$\lambda_y{}'(v) = \frac{\partial}{\partial t}\exp(tv)y\exp(-tv)\bigg|_{t=0} = vy - yv = [v, y],$$

thus (6) is also a special case of (5), and Algorithm 3 can be used also here.

4.4 Integration Methods Based on Frames

In this section we intend to show that the methods based on frames fit in the formalism of homogeneous spaces introduced in Sect. 3.

Let $\mathfrak{g}$ be the Lie algebra generated by the frame $\{E_1, \ldots, E_d\} \subset \mathfrak{X}(\mathcal{M})$ and let $\mathrm{G} \subset \mathrm{Diff}(\mathcal{M})$ be the collection of flows on $\mathcal{M}$ generated by exponentiating $\mathfrak{g}$. We define the action $\lambda : \mathrm{G}\times\mathcal{M} \to \mathcal{M}$ as

$$\lambda(\phi, y) = \phi(y).$$

We wish to solve

$$y' = \sum_{i=1}^{d} E_i f_i(y) = F_y(y), \tag{7}$$

[3] The set $\mathrm{Sym}(n)$ is a manifold, but not a Lie group.

where $F_y \in \mathfrak{g}$ is the vectorfield of frozen coefficients in (1). The map

$$F_p : \mathcal{M} \to \mathfrak{g}, \qquad p \mapsto F_p,$$

is of the same form as f in (5). Since

$$\lambda_y{}'(F_p) = \frac{\partial}{\partial t} \exp\left(tF_p\right)(y)\bigg|_{t=0} = F_p(y),$$

it follows that (7) can be written as

$$y' = \lambda_y{}'\left(F_y\right),$$

which is of the general form (5). This shows that Algorithm 3 can also be interpreted in this context. The central equations become

$$\begin{aligned} k_i &= f\left(\lambda(\exp(\tilde{u}_i), y_n)\right) = F_{\exp(\tilde{u}_i)(y_n)} \\ y_{n+1} &= \lambda\left(\exp(\tilde{v}), y_n\right) = \exp(\tilde{v})(y_n), \end{aligned}$$

and the brackets such as $[I_1, u_i]$ are Lie–Jacobi brackets between vectorfields.

4.5 Partial Differential Equations of Evolution

Several important partial differential equations of evolution arising in Physics can be written as a differential equation on (a subgroup of) the Lie group $\mathrm{Diff}(\mathcal{M})$. Given $\psi, \phi \in \mathrm{Diff}(\mathcal{M})$, we let $\lambda(\psi, \phi) = \psi \circ \phi$. Therefore,

$$\lambda_\phi{}'(f) = \frac{\partial}{\partial t} \exp\left(tf\right)(y)\bigg|_{t=0} \circ \phi = f \circ \phi,$$

where $f \in \mathfrak{X}(\mathcal{M})$. Thus (5) becomes

$$\phi_t{}' = f(\phi_t) \circ \phi_t.$$

The wave equation in elasticity, where ϕ is the configuration of the body (departure from initial position), the Poisson–Vlasov equations in plasma physics and the Euler flows in fluid dynamics are examples that fit naturally into the above description. The Euler flows evolve in the subgroup of volume-preserving diffeomorphisms on $\mathcal{M}$. A thorough treatment of these examples can be found in [9, 10]. It is possible to interpret Algorithm 3 in this context, although the numerical consequences of this interpretation is much more subtle. Since we cannot represent an infinite dimensional Lie group exactly, both the bracket and the exponential map have to be approximated. One procedure is, for instance, to represent $\mathfrak{X}(\mathcal{M})$ on a grid, and in this light the Lie-group approach becomes a version of the well known *method of lines,* where the spatial discretization is hidden for the time integration routine.

Comparing the Lie-group approach and the method of lines, we may gain important insight into strategies for discretizing partial differential equations, in such a way that important qualitative features are preserved (e.g. volume preservation for Euler flows). Secondly, it follows that the time integration can be performed independently of the spatial discretization. This suggests an object-oriented approach to integration of PDEs, whereby we can hide specific spatial discretization techniques within some appropriate classes.

5 Abstract RK Methods

One of our initial motivations for studying integration on manifolds, was an investigation of object-oriented programming techniques applied to the field of numerical computing. The main issue in object-oriented programming is the separation between 'what' and 'how', in other words, between *specification* and *implementation.* Formal specifications are a tool for finding a modularization of the program into classes. Ideally, the specification of a class should express a mathematical structure independently of a particular implementation. The implementation itself should be hidden within the class so that it can be later replaced with another implementation without doing modifications to the other classes. A numerical discretization is a matter of choosing an implementation, and it belongs to the 'how'-part of the programming. If we further follow this line of thought, we may conclude that also the choice of coordinate systems for a differential equation is a matter of 'how' rather than 'what'. A consequence of this philosophy is that it is important to search for *coordinate-free numerical techniques* [14], i.e. numerical algorithms that can be expressed independently of particular coordinate systems.

The examples we have presented in this paper display that RK-type methods may be expressed in an abstract coordinate-free manner involving the following basic structures:

- A Lie algebra defined as a vector space with a Lie bracket $[\_,\_]$.
- A Lie group G with a multiplication rule.
- An exponential map $\exp : \mathfrak{g} \to \mathrm{G}$.
- A homogeneous space $\mathcal{M}$ and an action $\lambda : \mathrm{G} \times \mathcal{M} \to \mathcal{M}$.
- A differential equation (5) defined by a function $f : \mathcal{M} \to \mathfrak{g}$.

Using modern computer languages (for instance C++), it is possible to program the time integration process in this abstract manner, such that particular implementations of these objects are subject to change.

A question is whether all the elements of this abstraction are necessary for expressing RK-type time integration schemes. The question is, at present, without a precise answer. However the principal component of the abstraction is that we have a domain $\mathcal{M}$ equipped with some basic continuous movements that can take us in all possible directions from every point. It seems difficult to give meaning to RK-type methods if we do not have such a structure.

6 Concluding Remarks

It is our feeling that this problem of integrationg differential equations on manifolds is becoming more (and better) understood. We believe that the Lie-group approach is indeed a big steep in this direction. Yet the basic ideas that we have presented are still in early stage and devote further analysis. Many other issues, like stability, and error control etc. need to be addressed in the future work.

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